Constitutive relation for "Newtonian fluids"

Fluid particle deformation will create viscous stresses.

$$\bar{\bar{\tau}} = -p\underbrace{I}_{\text{unit matrix}} + \underbrace{\bar{\bar{\sigma}}}_{\text{viscous stress tensor}} \qquad \tau_{ij} = -p\delta_{ij} + \sigma_{ij}$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \rightarrow \text{unit matrix}$$

$$\begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$$

↳ notations vary

For normal air, water, etc, the relation between viscous stress and deformation rate is linear. Also, these fluids have no inherent directional preference.

Rotation will not create viscous stresses

Look first in principal axis system of $\bar{\bar{\tau}}$. Here S must be diagonal too, or there would be a directional preference.

$$\bar{\bar{\tau}}' = \begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{pmatrix} \qquad S' = \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{pmatrix}$$

Relation must be of the form

$$\tau_1 = \overbrace{C_0}^{\text{must be } -p} + \overbrace{C_1}^{\text{call it } 2\mu+\lambda} s_1 + \overbrace{C_2}^{\text{call it } \lambda} s_2 + C_2 s_3$$

$$\tau_2 = C_0 + C_2 s_1 + C_1 s_2 + C_2 s_3$$

$$\tau_3 = C_0 + C_2 s_1 + C_2 s_2 + C_3 s_3$$

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial z'} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\tau_1 = -p + 2\mu s_1 + \lambda(s_1 + s_2 + s_3)$$

$$\cancel{\tau_1 = -p + \lambda(s_1+s_2+s_3) +}$$

$$s_1 + s_2 + s_3 = \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial z'} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$= \nabla \cdot \vec{v} \equiv \operatorname{div} \vec{v}$$ "the divergence of $\vec{v}$" (zero for incompressible fluids)

$$\tau_1 = -p + \lambda \operatorname{div}\vec{v} + 2\mu s_1$$ → primes not needed

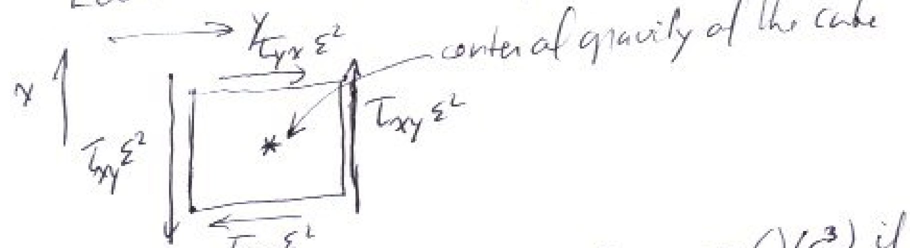
$$\bar{\bar{\tau}}^{(i)} = (-p + \lambda \operatorname{div}\vec{v}) + 2\mu S^{(i)} \rightarrow \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\boxed{\tau_{ij} = -p\,\delta_{ij} + \lambda \operatorname{div}(\vec{v})\,\delta_{ij} + 2\mu s_{ij}}$$

Stokes hypothesis: $\lambda = 2\mu$

Momentum Balance

Look at little cube of fluid with sides ε



$\vec{B}$: magnetic field

moment by stresses $\varepsilon^3\tau_{xy} - \varepsilon^3\tau_{yx} = O(\varepsilon^3)$ if $\tau_{xy} \neq \tau_{yx}$

angular momentum of fluid motion (as a continuum)

$$\int \vec{r}\times\vec{v}\,dm = O(\varepsilon^5) \text{: cannot survive for } \varepsilon \to 0$$

moment by Lorentz force $q\vec{v}\times\vec{B}$

$$\int \vec{r}\times\vec{v}\times\vec{B}\,dq = O(\varepsilon^5) \text{: cannot survive}$$

Conclusion: $\tau_{xy} = \tau_{yx}$ if all this is true: symmetric $\bar{\bar{\tau}}$

However, "continuum" fluid is really made up of molecules or atoms, and these have internal angular momentum, "spin".

$[\int \vec{r}\times\vec{v}\,dm]_{\text{molecule } i} = \vec{s}_i$, say (order $\hbar$)

Each molecule experiences a Lorentz moment of order $\frac{q}{m} s_i \cdot B$.

The number such moments is proportional to the number of molecules, hence to ε^3. Normally these moments are in all directions and cancel out. But if the "fluid" is ferromagnetic, they have a preferred direction and produce a moment $O(\varepsilon^3)$

$\Rightarrow \tau_{xy} \neq \tau_{yx}$.

Also the angular momentum of the molecules is ~~propo~~ $O(\varepsilon^3)$ too if they are preferentially aligned.

Unfortunately, in a real fluid with normal heat motion it would be hard to have significant alignment. In a solid like iron, alignment is really due to electrostatic energy minimalization in the presence of exclusion constraints.