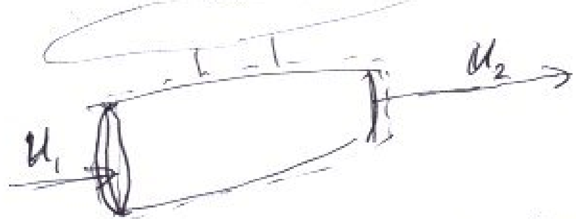
"Control Volume" or "Fixed Region" or "Arbitrary Region"

Not a material region

e.g. rocket

not a chunk of fluid → fluid is lost

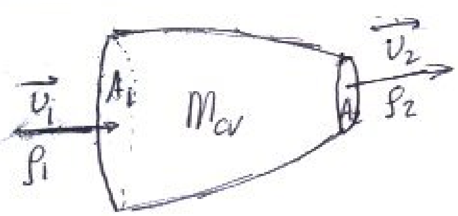
e.g. jet engine



not a chunk of fluid: contains a different chunk of fluid at different times

$\boxed{\text{For a control volume, must correct for in and out flow of fluid from the volume}}$

For a (simple) duct situation



mass outflow per unit time

"$\dot{m}_2$" $= \rho_2 |\vec{v_2}| A_2 = \rho_2 V_2 A_2$ if $V_2 = |\vec{v_2}|$

mass inflow per unit time

"$\dot{m}_1$" $= \rho_1 |\vec{v_1}| A_1 = \rho_1 V_1 A_1$

mass inside $M_{cv} = \int_{V_{cv}} \rho \, dV$

$$\frac{DM_{cv}}{Dt} = \text{inflow} - \text{outflow} = \dot{m}_1 - \dot{m}_2$$

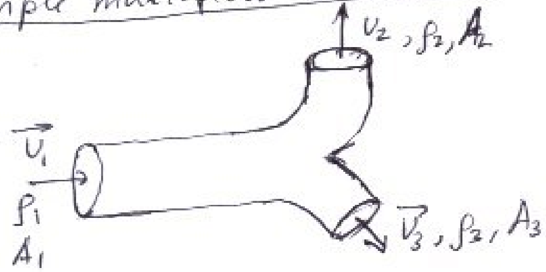
If M_{cv} is constant: inflow = outflow

simple duct filled with fluid, incompressible fluid $\} \; A_1 V_1 = A_2 V_2$

simple duct steady flow $\rho_1 A_1 V_1 = \rho_2 A_2 V_2$

} Simple duct continuity equation

Simple multiflow situations



replace "inflow" by "sum of inflows"
and "outflow" by "sum of outflows"

Nonsimple situations

The situation is nonsimple if for any through-flow area A_i
- $\vec{v}_i$ is not normal to A_i
- $|\vec{v}_i|$ is not the same at all points of A_i
- ρ_i " " " " " " "

For nonsimple situations:
inflow $\equiv$ negative outflow → no more "inflow"

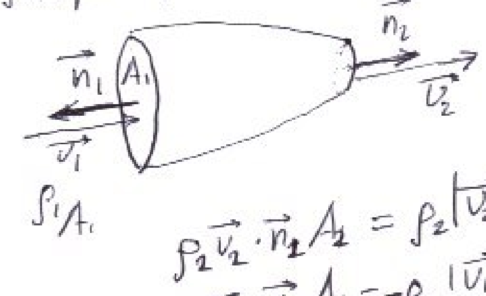
$$\frac{dM_{cv}}{dt} + \text{sum of the outflows} = 0$$

General formula for an area dA that is small enough that ρ and $\vec{v}$ are constant on it:

$$\boxed{\text{outflow through } dA: \text{ "}\dot{dm}\text{"} = \rho\, \vec{v} \cdot \vec{n}\, dA}$$

where $\vec{n}$ is the unit vector normal to dA sticking out of the control volume (duct)

Compare for simple flow



$$\rho_2 \vec{v}_2 \cdot \vec{n}_2 A_2 = \rho_2 |\vec{v}_2| A_2$$

$$\rho_1 \vec{v}_1 \cdot \vec{n}_1 A_1 = -\rho_1 |\vec{v}_1| A_1$$

So $\dfrac{dM_{cv}}{dt} + \rho_2|\vec{v}_2|A_2 - \rho_1|\vec{v}_1|A_1 = 0$: same as before

$$\boxed{\frac{dM_{cv}}{dt} + \int_{\text{all } A_{cv}} \rho\, \vec{v} \cdot \vec{n}\, dA = 0} \quad \text{General continuity equation}$$

Note: if A is moving, replace $\vec{v}\cdot\vec{n}$ by $(\vec{v}-\vec{v}_A)\cdot\vec{n}$

$\vec{v}$: fluid velocity; $\vec{v}_A$: surface velocity

(That is why there is no surface integral for a material region: ~~for~~ since a material region moves with the flow, $\vec{v}_A = \vec{v}$.)

(The fact that derivatives of integrals with moving boundaries (here relative to the fluid) have surface integrals is known as the "Leibniz theorem")

Momentum equation:

Material region: $\dfrac{D\vec{l}_{mR}}{Dt} = \vec{F}_{surface} + \vec{F}_{body}$

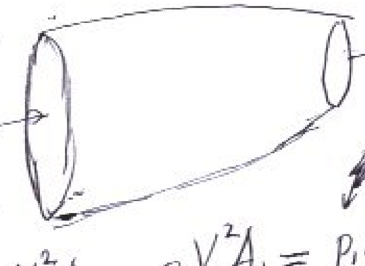
$\rho\vec{v}\cdot\vec{n}\,\vec{v}\,dA$ is the momentum flow through A

For a control volume

$$\boxed{\frac{d\vec{l}_{cv}}{dt} + \int_{all A_{cv}} \rho\,\vec{v}\cdot\vec{n}\,\vec{v}\,dA = \vec{F}_{surface} + \vec{F}_{body}} \qquad \vec{l}_{cv} = \int_{cv}\rho\vec{v}\,d\forall$$

If A is moving, replace $\vec{v}\cdot\vec{n}$ by $(\vec{v}-\vec{v}_A)\cdot\vec{n}$ but leave the second $\vec{v}$ alone.

Example for simple duct in steady flow



$\vec{v}_2\cdot\vec{n} = +|\vec{v}_2| = V_2$

$\Downarrow \vec{g}$

$\vec{v}_1\cdot\vec{n} = -|\vec{v}_1| = -V_1$

x component of momentum

$$\cancel{\frac{d l_{cv_x}}{dt}}^{\,steady} + \rho_2 V_2^2 A_2 - \rho_1 V_1^2 A_1 = p_1 A_1 - p_2 A_2 + F_{x\ \text{pipe surface on fluid}}$$

assumes there is no viscous stress at A_1 and A_2 (at least in the x direction: $\tau_{xx} = 0$)