Note: if A is moving, replace $\vec{v}\cdot\vec{n}$ by $(\vec{v}-\vec{v}_A)\cdot\vec{n}$
($\vec{v}$: fluid velocity; $\vec{v}_A$: surface velocity)

(That is why there is no surface integral for a <u>material region</u>: ~~for~~ since a material region moves with the flow, $\vec{v}_A = \vec{v}$.)

(<u>The</u> fact that derivatives of integrals with moving boundaries (here relative to the fluid) have surface integrals is known as the "Leibniz theorem", ~~&~~ or Reynolds theorem) ~~for mat..~~ → definitions vary

<u>Momentum equation:</u>

Material region: $\dfrac{D\vec{l}_{mr}}{Dt} = \vec{F}_{surface} + \vec{F}_{body}$

For a control volume ($\rho\,\vec{v}\cdot\vec{n}\,\vec{v}\,dA$ is the momentum flow through A)

$$\frac{d\vec{l}_{cv}}{dt} + \int_{all A_{cv}} \rho\,\vec{v}\cdot\vec{n}\,\vec{v}\,dA = \vec{F}_{surface} + \vec{F}_{body}$$

$$\vec{l}_{cv} = \int_{\forall_{cv}} \rho\,\vec{v}\,d\forall$$

If A is moving, replace $\vec{v}\cdot\vec{n}$ by $(\vec{v}-\vec{v}_A)\cdot\vec{n}$ but leave the second $\vec{v}$ alone.

Example for simple duct in steady flow

$\rho_2 A_2$, $\vec{v}_2$: $\vec{v}_2\cdot\vec{n} = +|\vec{v}_2| = V_2$

$\rho_1 A_1$, $\vec{v}_1$: $\vec{v}_1\cdot\vec{n} = -|\vec{v}_1| = -V_1$

$\Downarrow \vec{g}$

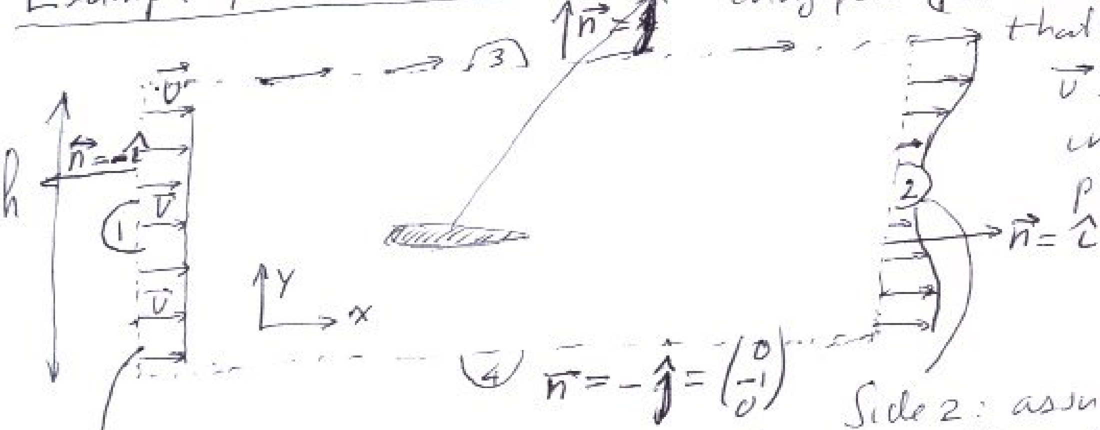
x component of momentum (steady):

$$\frac{d l_{cv_x}}{dt} + \rho_2 V_2^2 A_2 - \rho_1 V_1^2 A_1 = p_1 A_1 - p_2 A_2 + F_{x\ \text{pipe surface on fluid}}$$

(assumes there is no viscous stress at A_1 and A_2 (at least in the x direction: $\tau_{xx} = 0$)

Example 4.1 from Kundu

2D body, for which we want to know the drag force (per unit span) in a stream that has a constant velocity $\vec{v} = U\hat{i}$, U constant, at infinity, and a constant pressure p_∞ at infinity



Side 1: assume that $v_x \approx U$ and $p \sim p_\infty$ → requires v_y negligible. OK if ① is far enough upstream

Side 2: assume that $v_x \approx U$ near the ends and $p \sim p_\infty$ → requires $v_y \sim 0$

Far enough downstream... but not so far that we cannot measure the dip in the velocity profile accurately.

②, ③ $v_x \approx U$
v_y small but not zero
$p \sim p_\infty$

Since the velocities are about constant on surface points, no viscous stresses on surface: pressure only stress acting

Mass conservation: (0 since steady)

$$\frac{DM_{cv}}{Dt} + \int_{A_a} \rho \vec{v}\cdot\vec{n}\, dA = 0.$$

$$= 0 + \underbrace{\int_{①} -\rho u \, 1 dy}_{+\dot{m}_{1\,out}} \;(\vec{v}\cdot\vec{n} = -u,\ dA) + \underbrace{\int_{②} +\rho u\, 1dy}_{\dot{m}_{2\,out}} \;(\vec{v}\cdot\vec{n},\ dA) + \underbrace{\int_{③} \rho v \, 1dx}_{\dot{m}_{3\,out}} \;(\vec{v}\cdot\vec{n},\ dA) - \underbrace{\int_{④} \rho v\, 1dx}_{\dot{m}_{4\,out}} \;(dA,\ \vec{v}\cdot\vec{n} = -v)$$

Momentum conservation (0 since steady)

$$\frac{DP_{cv\,x}}{Dt} + \int_{A_{cv}} \rho \vec{v}\cdot\vec{n}\, u\, dA = \int_{A_{cv}} (\overline{\overline{\tau}}\cdot\vec{n})_x\, dA + \int_{V_{cv}} \rho g_x\, dV - F_D/b$$

($g_x = 0$; F_D/b: drag force / unit span; b: unit meter)

inviscid: $\overline{\overline{\tau}} = -p\,I$, $\overline{\overline{\tau}}\,\vec{n} = -p\,\vec{n}$

$$(\overline{\overline{\tau}}\,\vec{n})_x = -p n_x = \begin{cases} 0 & \text{on ③ and ④} \\ -p & \text{on ②} \\ +p & \text{on ①} \end{cases}$$

$$\int_{1} -\rho \overset{u=U}{u} u\, 1\, dy + \int_{2} \rho u u\, 1\, dy + \int_{3} \rho v \overset{u=U}{u}\, 1\, dx + \int -\rho v \overset{u=U}{u}\, 1\, dx$$

equal and opposite

$$= \int_{1} p\, 1\, dy + \int_{2} -p\, 1\, dy + 0 + 0 + 0 - F_D/b$$

$p = p_\infty$ $\qquad$ $p = p_\infty$

mass conservation (continuity)

The integrals ③ and ④ can be simplified

$$U\int_{3} \rho v\, 1\, dx + U\int_{4} -\rho v\, 1\, dx = U(\dot{m}_{3out} + \dot{m}_{4out}) =$$

$$= U(-\dot{m}_{1out} - \dot{m}_{2out})$$

$$\dot{m}_{1out} \to +\int_{1} \rho U dy \qquad \dot{m}_{2out} \to -\int_{2} \rho u\, dy$$

$$-\rho U^2 \int_{1\,(2)} dy + \int_{2} \rho u^2 dy + U\left[\rho U \int_{1\,(2)} dy - \int_{2} \rho u\, dy\right] = -F_D/b$$

$$\frac{F_D}{b} = \int_{2} \rho (U-u)\, u\, dy$$

Note $\int \left(1 - \frac{u}{U}\right)\frac{u}{U} dy$ is called the "momentum thickness" Θ

$$\boxed{\frac{F_D}{b} = \rho U^2 \Theta}$$

It is like ~~a str~~ the momentum of a strip of thickness Θ of the incoming flow completely disappeared, and the rest kept all its momentum

Arbitrary Control Volume

$$M_{cv} = \int_{V_{cv}} \rho \, dV \qquad \vec{l}_{cv} = \int_{V_{cv}} \rho \vec{v} \, dV$$ as always

$$\frac{dM_{cv}}{dt} + \int_{A_{cv}} \rho (\vec{v} - \vec{v}_A) \cdot \vec{n} \, dA = 0$$

$$\frac{d\vec{l}_{cv}}{dt} + \int_{A_{cv}} \rho (\vec{v} - \vec{v}_A) \cdot \vec{n} \, \vec{v}_A \, dA = \vec{F}$$

$\vec{F} \rightarrow \vec{F}_{surface} + \vec{F}_{body}$

arbitrary CV

Rigid, translating control volume

$\vec{v}_A$ is the same at all points on the surface so

$$\int \rho (\vec{v} - \vec{v}_A) \cdot \vec{n} \, dA = \int \rho (\vec{v} - \vec{v}_A) \cdot \vec{n} \, (\vec{v} - \vec{v}_A) \, dA + \underbrace{\int \rho (\vec{v} - \vec{v}_A) \cdot \vec{n} \, dA}_{-\frac{dM_{cv}}{dt}} \, \vec{v}_A$$

$$\vec{l}_{cv} = \int \rho \vec{v} \, dV = \int \rho (\vec{v} - \vec{v}_A) \, dV + \int \rho \, dV \, \vec{v}_A$$

$\vec{v} - \vec{v}_A$ is called the "relative velocity" $\vec{v}_{rel}$

$$\vec{l}_{cv_{rel}} = \int_{V_{cv}} \rho \vec{v}_{rel} \, dV \qquad M_{cv} = \int_{V_{cv}} \rho \, dV$$

$$\frac{dM_{cv}}{dt} + \int_{A_{cv}} \rho \vec{v}_{rel} \cdot \vec{n} \, dA = 0$$

$$\frac{d\vec{l}_{cv_{rel}}}{dt} + M_{cv} \frac{d\vec{v}_A}{dt} + \int_{A_{cv}} \rho \vec{v}_{rel} \cdot \vec{n} \, \vec{v}_{rel} \, dA = \vec{F}$$

rigid, ~~non~~ translating CV

Angular momentum conservation for a CV

See an undergraduate fluids book