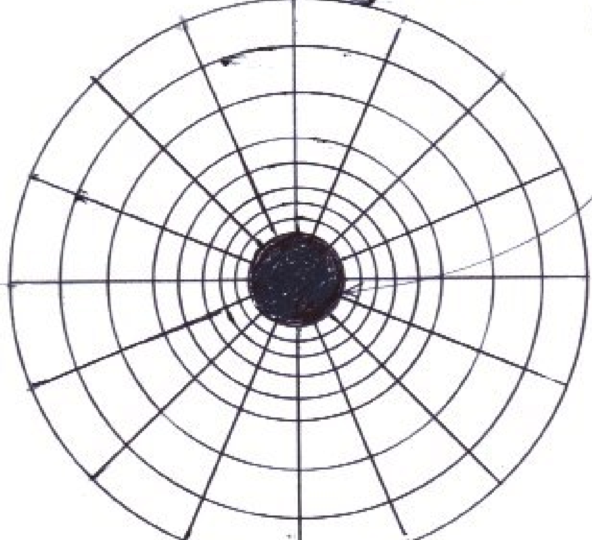
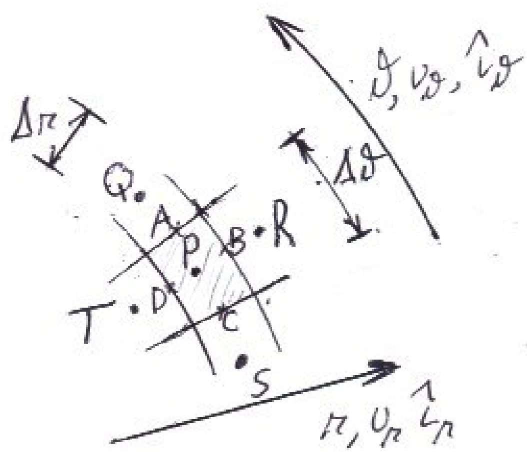
Idea of the Finite volume method (Panton 5.7)

- Chop flow region up into many small pieces, called "finite elements".
- For each "finite element", write approximate (mass, momentum, [energy]) conservation equations
- Add boundary conditions at the wall, "infinity"
- Solve the resulting large system of equations



Example: mass conservation for a small finite element in polar coordinates, Flow is 2D, incompressible,

- Our volumes are as shown above.
- Our unknowns are the values of $\xrightarrow{\text{2D: zero}}$
[ρ] v_r, v_θ, [v_z], p
incompressible → known
at the centers of the volumes (i.e. at P, Q, R, S, and T in the sketch right)
- Our equations are continuity, x-momentum, y momentum, [z-momentum], [energy] for each finite volume



A, B, C, and D are the center points of the sides a, b, c, d.
Their values of ρ, v_r, v_θ are ~~not computed, not unknowns~~.

So we have 3 unknowns for each control volume and also 3 equations for each control volume: perfect!
(But the values of v_r, v_ϑ, p at A, B, C, and D are not unknowns)
Here: write continuity for a typical element as sketched above

HW: write x-momentum for the same

Exam: maybe equivalent for a Cartesian control volume??
(if I did not yet do all variations)

Note: finite element is approximately a small rectangle with sides Δr and $r\Delta\vartheta$
$\vec{n}$ at point B (center point outer side): $\hat{\imath}_r$
$\vec{n}$ at point A (largest ϑ): $\hat{\imath}_\vartheta$

$$\frac{dM_{cv}}{dt} + \int_{A_{cv}} \rho \vec{v}\cdot\vec{n}\, dA = 0 \qquad M_{cv} = \int_{V_{cv}} \rho\, dV$$

A_{cv} consists of the four line segments bordering the element, i.e. the perimeter of the rectangle
V_{cv} is the area of the rectangle

Approximate at a given time t, next time $t+\Delta t$

$$M_{cv} \approx \int_{V_{cv}} \rho\, dV \approx \rho_P \underbrace{r\Delta r\,\Delta\vartheta}_{V_{cv}}$$

$$M_{cv}(t+\Delta t) \approx \rho_P(t+\Delta t)\, r\,\Delta r\,\Delta\vartheta$$

$$\frac{dM_{cv}}{dt} \approx \frac{\rho_P(t+\Delta t) - \rho_P}{\Delta t}\, r\,\Delta r\,\Delta\vartheta \qquad (= 0 \text{ for incompressible})$$

$$\int_a \rho\vec{v}\cdot\overset{\hat{\imath}_\vartheta}{\vec{n}}\,\overset{\Delta r}{dA} \approx \rho_A \vec{v}_A\cdot\hat{\imath}_\vartheta\,\Delta r = \rho_A v_{\vartheta A}\,\Delta r$$

Approximate further, since ρ_A and $v_{\vartheta A}$ are not unknowns.
Take the average of $\rho_P v_{\vartheta P}$ and $\rho_Q v_{\vartheta Q}$:

$$\int_a \rho\vec{v}\cdot\vec{n}\,dA \approx \frac{\rho_P v_{\vartheta P} + \rho_Q v_{\vartheta Q}}{2}\,\Delta r$$

Similarly

$$\int_c \rho\vec{v}\cdot\vec{n}\,dA \approx -\frac{\rho_P v_{\vartheta P} + \rho_S v_{\vartheta S}}{2}\,\Delta r \qquad (\text{or } -\rho_C v_{\vartheta C}\,\Delta r)$$

since $\vec{n}$ is now $-\hat{\imath}_\vartheta$

$$\oint_B \rho \vec{v} \cdot \vec{n}\, dA \approx \rho_B v_{rB} r_B \Delta\vartheta \approx$$

$$\approx \frac{\rho_P v_{rP} r_P + \rho_R v_{rR} r_R}{2} \Delta\vartheta$$

$\vec{n} = \hat{i}_r$

$$\int_C \rho \vec{v} \cdot \vec{n}\, dA \approx \rho_D v_{rD} r_D \Delta\vartheta$$

$$\approx \frac{\rho_P v_{rP} r_P + \rho_T v_{rT} r_T}{2} \Delta\vartheta$$

Total continuity equation for the element

$$\frac{dM_{cv}}{dt} + \oint \rho \vec{v} \cdot \vec{n}\, dA = 0$$

zero for incompressible

$\rho_R = \rho_T = \rho_Q = \rho_S$
can be divided
out for incompressible

$$\frac{\rho_P(t+\Delta t) - \rho_P}{\Delta t} r \Delta r \Delta\vartheta$$

$$+ \frac{\rho_R v_{rR} r_R - \rho_T v_{rT} r_T}{2} \Delta\vartheta$$

$$+ \frac{\rho_Q v_{\vartheta Q} - \rho_S v_{\vartheta S}}{2} \Delta r = 0$$

Incompressible

$$\frac{v_{rR} r_R - v_{rT} r_T}{2} \Delta\vartheta + \frac{v_{\vartheta Q} - v_{\vartheta S}}{2} \Delta r = 0$$