Boundary conditions (Kundu 4.10, Panton 6.4)

1) At a solid surface that is at rest

1a) viscous: $\vec{v} = 0$ at the wall

"no slip"
"impenetrable"

1b) inviscid: $\vec{v}\cdot\vec{n} = 0$ at the wall

"slip"
"impenetrable", velocity tangential to the wall

2) At a solid surface that is moving:

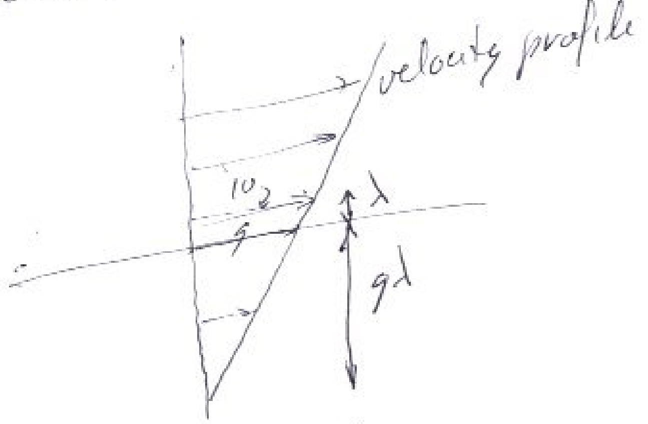
2a) viscous $\vec{v} = \vec{v}_{wall}$ at the wall

2b) inviscid $\vec{v}\cdot\vec{n} = \vec{v}_{wall}\cdot\vec{n}$

Molecular explanation for no-slip:

Molecules just above the wall hit the wall and lose at least some of their organized forward motion. By itself, that would mean that the flow would be slowed down just a bit, not that $\vec{v}=0$ at the wall. But there is a feedback mechanism. The slower velocity at the wall "eats" into the overlying flow due to viscous diffusion. So after a bit of time, the molecules hitting the wall are already slow before they hit the wall. So the feedback becomes stronger. In a bit, the average velocity near the wall becomes zero even for a quite smooth wall on molecular scales.

For example, assume that the molecules lose only 10% of their organized motion in collision with the wall. Ballparking the velocity at the wall as 10% less than the velocity 1 free path length above the wall, the nonzero velocity at the wall extrapolates to zero 9 free path lengths below the wall: 18B



Because 9λ is an extremely small distance (compared with boundary layer thickness), in most cases, practically speaking the velocity is zero at the wall rather than at a virtual point 9 free path lengths below the wall.

3) Viscous, porous permeable boundary arrest: $\vec{v} = V_{\text{[illegible]}} \vec{n}$

4) Heat flux: assuming Fourier's law, the heat flux is $\vec{q} = k\nabla T = k \operatorname{grad} T = k \begin{pmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{pmatrix}$

The heat flux out of a boundary is given by $\vec{q} = k\frac{\partial T}{\partial n}$

$\frac{\partial T}{\partial n} = (\nabla T)\cdot\vec{n}$ is derivative of T in the direction normal to the boundary.

So if there is no heat flux into the wall (nonconducting) $\frac{\partial T}{\partial n} = 0$

5) Temperature given at the boundary

Pointwise equations for fluid mechanics

Needed for all theoretical work and exact solutions.
And for understanding flow features.

Proper way to set the equations is from the integral laws (i.e. control volumes).

~~One, not so rigorous way~~

Method 1 (Not so rigorous): Take the approximate equation for a small finite element, then let the size of the element become zero

Example for continuity: finite element equation as found previously:

$$\frac{\rho_P(t+\Delta t)-\rho_P}{\Delta t}\, r\,\Delta r\,\Delta\vartheta + (\rho_B v_{rB} r_B - \rho_D v_{rD} r_D)\Delta\vartheta + (\rho_A v_{\vartheta A} - \rho_C v_{\vartheta C})\Delta r = 0$$

Divide by $r\,\Delta r\,\Delta\vartheta$

$$\frac{\rho_P(t+\Delta t)-\rho_P}{\Delta t} + \frac{1}{r}\frac{\rho_B v_{rB} r_B - \rho_D v_{rD} r_D}{\Delta r} + \frac{1}{r}\frac{\rho_A v_{\vartheta A} - \rho_C v_{\vartheta C}}{\Delta\vartheta} = 0$$

Definition: The derivative of a function $f(x)$ of some variable x is given by

$$\frac{df}{dx} = \lim_{\Delta x\to 0} \frac{\text{change in } f \text{ due to some change } \Delta x \text{ in } x}{\text{change in } x}$$

$$= \lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

So $\dfrac{\rho_P(t+\Delta t)-\rho_P}{\Delta t}$ becomes $\dfrac{\partial\rho}{\partial t}$ in the limit $\Delta t \to 0$!

(∂ instead of d, since this derivative keeps ~~and~~ position constant, i.e. r and ϑ are not changed)

Similarly.

$$\frac{\rho_B v_{rB} r_B - \rho_D v_{rD} r_D}{\Delta r} \longrightarrow \frac{\partial \rho v_r r}{\partial r}$$ 20

$$\frac{\rho_A v_{\vartheta A} - \rho_C v_{\vartheta C}}{\Delta \vartheta} \longrightarrow \frac{\partial \rho v_\vartheta}{\partial \Delta \vartheta}$$

Total differential continuity equation

$$\frac{\partial \rho}{\partial t} + \underbrace{\frac{1}{r}\frac{\partial \rho v_r r}{\partial r} + \frac{1}{r}\frac{\partial \rho v_\vartheta}{\partial \vartheta}}_{\text{table book or appendix B: } = \operatorname{div}(\rho \vec{v})} = 0$$

Apparently then

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0}$$ is differential continuity equation

Method 2 Rigorous, and quicker: convert surface integrals into volume ones, using divergence theorem

divergence theorem: If V is a volume and A is the complete terminating surface of V, and $\vec{v}$ any vector:

$$\boxed{\int_V \operatorname{div} \vec{v} \, . \, dV = \oint_A \vec{v} \cdot \vec{n} \, dA}$$

Index notation

$$\int_V \frac{\partial v_i}{\partial x_i} \, dV = \oint_A v_i n_i \, dA$$

But summation is not needed:

$$\boxed{\int_V \frac{\partial f}{\partial x_i} \, dV = \oint_A f n_i \, dA}$$

i.e. going from surface integrals to volume integrals:

$n_i \rightarrow \frac{\partial}{\partial x_i}$ [entire rest of the integrand]

$dA \Rightarrow dV$