Similarly.

$$\frac{\rho_B v_{rB} r_B - \rho_D v_{rD} r_D}{\Delta r} \longrightarrow \frac{\partial \rho v_r r}{\partial r}$$

$$\frac{\rho_A v_{\vartheta A} - \rho_C v_{\vartheta C}}{\Delta \vartheta} \longrightarrow \frac{\partial \rho v_\vartheta}{\partial \Delta \vartheta}$$

Total differential continuity equation

$$\frac{\partial \rho}{\partial t} + \underbrace{\frac{1}{r}\frac{\partial \rho v_r r}{\partial r} + \frac{1}{r}\frac{\partial \rho v_\vartheta}{\partial \vartheta}}_{\text{table book or appendix B:}} = 0 \qquad = \operatorname{div}(\rho \vec{v})$$

Apparently then

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0}$$ is differential continuity equation

Method 2 Rigorous, and quicker: convert surface integrals into volume ones, using divergence theorem

divergence theorem: If $\forall$ is a volume and A is the complete terminating surface of $\forall$, and $\vec{v}$ any vector:

$$\boxed{\int_{\forall} \operatorname{div} \vec{v} \, d\forall = \oint_A \vec{v} \cdot \vec{n} \, dA}$$

Index notation

$$\int_{\forall} \frac{\partial v_i}{\partial x_i} d\forall = \oint_A v_i n_i \, dA$$

But summation is not needed:

$$\boxed{\int_{\forall} \frac{\partial f}{\partial x_i} d\forall = \oint_A f n_i \, dA}$$

i.e. going from surface integrals to volume integrals:

$$n_i \rightarrow \frac{\partial}{\partial x_i} \text{[entire rest of the integrand]}$$

$$dA \Rightarrow d\forall$$

Apply to continuity:

$$\frac{d M_{cv}}{dt} + \int_{A_{cv}} \rho \vec{v}\cdot\vec{n}\, dA = 0 \quad \xrightarrow{\text{or}} \quad \frac{d\int_{V_{cv}} \rho dV}{dt} + \int_{A_{cv}} \rho \vec{v}\cdot\vec{n}\, dA = 0$$

$$\frac{d\int_{V_{cv}} \rho\, dV}{dt} + \int_{A_{cv}} \rho\, v_i n_i\, dA = 0 \qquad \int_{V_{cv}} \frac{\partial \rho}{\partial t} dV + \int_{V_{cv}} div(\rho\vec{v}) dV = 0$$

change order of differential & integration ↓ div theorem

$$\int_{V_{cv}} \frac{\partial \rho}{\partial t} dV + \int_{V_{cv}} \frac{\partial \rho v_i}{\partial x_i} dV = 0 \qquad \int_{V_{cv}} \left[\frac{\partial \rho}{\partial t} + div(\rho\vec{v})\right] dV = 0$$

$$\int_{V_{cv}} \left[\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i}\right] dV = 0$$

this can only be true for <u>any</u> control volume V_{cv} if the integrand is <u>everywhere</u> 0

$$\boxed{\frac{\partial \rho}{\partial t} + \underbrace{\frac{\partial \rho v_i}{\partial x_i}}_{div\, \rho\vec{v} \text{ in index notation}} = 0}$$

differential continuity in conservation form

Apply to momentum

$$\frac{d\vec{l}_{cv}}{dt} + \int \rho\vec{v}\cdot\vec{n}\,\vec{v}\, dA = \int \overline{\overline{\tau}}^{T}_{vis}\, \vec{n}\, dA - \int p\vec{n}\, dA + \int \rho\vec{g}\, dV$$

make sure to use unique index names!

$$\frac{d\int \rho v_i\, dV}{dt} + \int \rho v_j n_j v_i\, dA = \int \tau^{v}_{ji} n_j\, dA - \int p n_i\, dA + \int \rho g_i\, dV$$

$$\int \frac{\partial \rho v_i}{\partial t} dV + \int \frac{\partial \rho v_j v_i}{\partial x_j} dV = \underbrace{\int \frac{\partial \tau^{v}_{ji}}{\partial x_j} dV - \int \frac{\partial p}{\partial x_i} dV + \int \rho g_i\, dV}_{\text{Force per unit volume}}$$

$$\boxed{\frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_j v_i}{\partial x_j} = \frac{\partial \tau^{visc}_{ji}}{\partial x_j} - \frac{\partial p}{\partial x_i} + \rho g_i}$$

differential momentum in conservation form

inviscid: $\frac{\partial \tau^{visc}_{ji}}{\partial x_j} = 0$

Apply to energy:

Let internal energy per unit mass be e

Kinetic energy per unit mass is $\frac{1}{2}V^2 = \frac{1}{2}|\vec{V}|^2$

Treat gravity as a force doing work, rather than an energy per unit mass and height.

$$E_{CV} = \int_{V_{CV}} \rho\left(e + \tfrac{1}{2}V^2\right) dV$$

work done by pressure forces on surface

$$\frac{dE_{CV}}{dt} + \int_{A_{CV}} \rho \vec{V}\cdot\vec{n}\left(e + \tfrac{1}{2}V^2\right) dA \cancel{+ \int p \vec{n}\cdot\vec{V}\,dA - \int \vec{\vec{\tau}}^{vT}\vec{n}\cdot\vec{V}\,dA}$$

$$= -\int_{A_{CV}} p\, \vec{V}\cdot\vec{n}\,dA + \int_{A_{CV}} \vec{V}\cdot \vec{\vec{\tau}}^{vT}\vec{n}\,dA + \int \rho\, \vec{V}\cdot\vec{g}_i\,dV$$

work done by pressure and viscous forces on surface A_{CV} per unit time

work done by gravity on fluid particles

$$\boxed{\frac{\partial \rho\left(e + \frac{1}{2}v_i v_i\right)}{\partial t} + \frac{\partial \rho v_j\left(e + \frac{1}{2}v_i v_i\right)}{\partial x_j} = -\frac{\partial p v_j}{\partial x_j} + \frac{\partial \tau_{ji}^{visc} v_i}{\partial x_j} + \rho v_i g_i}$$