Lagrangian versus Eulerian

Physics: Classical physics is typically Lagrangian; material particle based.

For example

Particle: mass m = constant

$$\text{position } \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{r}(t)$$

$$\vec{v} = \frac{D\vec{r}}{Dt} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad \vec{a} = \frac{D\vec{v}}{Dt}$$

$$m\vec{a} = \vec{F}$$

Equivalent ~~Lagr~~ Lagrangian description in fluids:

Infinitesimal particle with volume $D\mathcal{V}$

$$\text{mass } \quad Dm = \rho\, D\mathcal{V} = \text{constant} \quad \text{(but } D\mathcal{V} \text{ might not be)}$$

$$\text{position } \boxed{\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{r}(t;\ \text{particle})}$$

$$\vec{v} = \frac{D\vec{r}}{Dt} \qquad \vec{a} = \frac{D\vec{v}}{Dt}$$

$$D\vec{F} = \vec{a}\,Dm = \vec{g}\,Dm + D\vec{F}_p + D\vec{F}_{\text{viscous}}$$

Typically, the particle label is chosen to be the initial position $\vec{r}\big|_{t=0} \equiv \vec{r}_0$ of the particle.

Then $\boxed{\vec{r} = \vec{r}(t, \vec{r}_0) \quad \text{Lagrangian description}}$

Unfortunately, using particles in fluids is usually awkward and inefficient. Particles get lost from the region of interest as they go downstream

Eulerian description in Fluids

Based on "fields", in which the density, pressure, velocity, ... are taken to be mathematical functions of position and time, rather than particle label and time

e.g. $p = p(\vec{r}, t) \qquad \vec{v} = \vec{v}(\vec{r}, t)$

Unfortunately, often you are not just interested in total pressure and velocity at a point are (regardless of what particle it is.) Often you are interested in what the acceleration of the particle at the point is, or its change in temperature in time, say.

Have $\vec{v}(r,t)$ ~~but normall~~ but

$$\boxed{\frac{\partial \vec{v}}{\partial t} \neq \frac{D\vec{v}}{Dt} \text{ in general}}$$

$\frac{\partial \vec{v}}{\partial t}$ is a mathematical quantity, not a particle acceleration at that point

~~Counter~~ Example: flow through a pipe; incompressible and steady:

$V_2 > V_1$ since $A_2 < A_1$

$$AV_1 = AV_2 = AV = \text{constant}$$

Steady: $\frac{\partial \vec{v}}{\partial t} = 0$, but the particles accelerate in the pipe (pick up speed): $\frac{D\vec{v}}{Dt} > 0$

Key formula: Lagrangian / Material / Substantial derivative

$$\boxed{\begin{aligned}
\frac{Df}{Dt} &= \frac{\partial f}{\partial t} + (\vec{v}\cdot\nabla)f \\
&= \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} && \text{Cartesian} \\
&= \frac{\partial f}{\partial t} + v_r\frac{\partial f}{\partial r} + \frac{v_\theta}{r}\frac{\partial f}{\partial \theta} + v_z\frac{\partial f}{\partial z} && \text{Cylindrical, scalar } f \\
&= \frac{\partial f}{\partial t} + v_r\frac{\partial f}{\partial r} + \frac{v_\theta}{r}\frac{\partial f}{\partial \theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial f}{\partial \varphi} && \text{Spherical scalar } f
\end{aligned}}$$

This is just the total derivative of Calculus

$$df = \frac{\partial f}{\partial t}dt + \underbrace{\frac{\partial f}{\partial x}\underset{\downarrow}{dx}_{\,u\,dt} + \frac{\partial f}{\partial y}\underset{\downarrow}{dy}_{\,v\,dt} + \frac{\partial f}{\partial z}\underset{\downarrow}{dz}_{\,w\,dt}}_{\text{if particle}}$$

Then $\vec{a} = \frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + u\frac{\partial \vec{v}}{\partial x} + v\frac{\partial \vec{v}}{\partial y} + w\frac{\partial \vec{v}}{\partial z}$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

Put a few markers in the fluid and take a long time exposure

Flow lines

Particle paths: $\frac{D\vec{x}}{Dt} = \vec{v}$

so

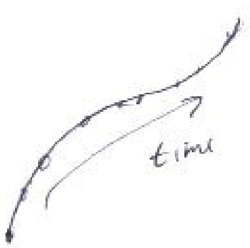
$$\boxed{\begin{aligned} &\frac{Dx}{u} = \frac{Dy}{v} = \frac{Dz}{w} = Dt \quad \text{Cartesian} \\ &\frac{Dr}{v_r} = \frac{r D\vartheta}{v_\vartheta} = \frac{Dz}{v_z} = Dt \quad \text{Cylindrical} \\ &\frac{Dr}{v_r} = \frac{r D\vartheta}{v_\vartheta} = \frac{r \sin\vartheta D\varphi}{v_\varphi} = Dt \quad \text{spherical} \end{aligned}}$$

Particle Paths

To integrate, use the solvable equalities. given $\vec{v}$

E.g. you can solve $\frac{Dx}{u} = \frac{Dy}{v}$ only if

z does not occur or is known as a function of x and y



Put a lot of markers in the fluid and take a short time exposure

Streamlines: $d\vec{r} \,//\, \vec{v}$ everywhere parallel to the velocity

$dt = 0$ at a given, single time

single time

so

$$\boxed{\begin{aligned} &\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad dt=0 \quad \text{Cartesian} \\ &\frac{dr}{v_r} = \frac{r d\vartheta}{v_\vartheta} = \frac{dz}{v_z} \quad dt=0 \quad \text{Cylindrical} \\ &\frac{dr}{v_r} = \frac{r d\vartheta}{v_\vartheta} = \frac{r\sin\vartheta d\varphi}{v_\varphi} \quad dt=0 \quad \text{Spherical} \end{aligned}}$$

Streamlines

Streaklines: all pathlines that have come out of a smoke generator at position $\vec{r}_0$ seen at a given time t (Smoke stack)

Streaklines can be obtained by finding the position $\vec{r}(t,\tau)$ of the particle coming out of the generator at

Streaklines can be found by finding the position $\vec{r}(t,\tau)$ of a generic particle coming out of the generator at some arbitrary time τ. Then eliminate τ to, say, find y and z as functions of x.

Examples: Panton $\vec{v} = cx\hat{i} - cy\hat{j}$ (Ideal stagnation point flow)

Kundu "water waves"

We: waterwaves

$u = \varepsilon \sin(lx + \omega t)$ $\qquad l, \omega, \varepsilon > 0 \qquad \varepsilon$ small

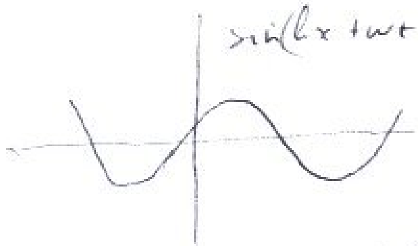
$v = -\varepsilon \cos(lx + \omega t)$

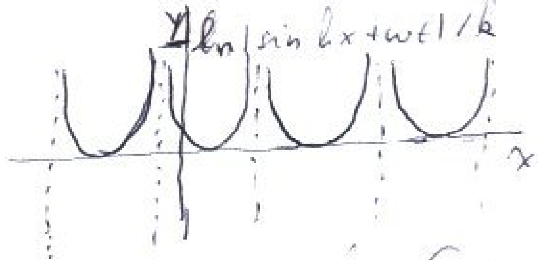
Streamlines

$$dx : dy : dt = u : v : 0 \quad \rightarrow \quad \frac{dy}{dx} = \frac{v}{u} = -\frac{\cos(lx+\omega t)}{\sin(lx+\omega t)}$$

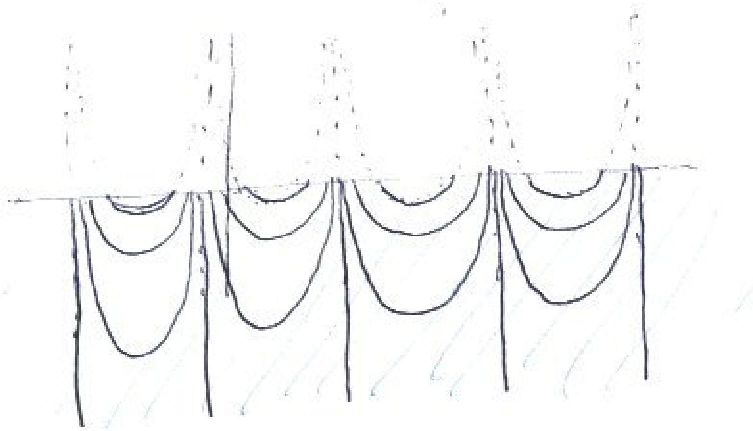
$$\rightarrow \quad dy = -\frac{\cos(lx+\omega t)}{\sin(lx+\omega t)}\, dx = -\frac{1}{l}\, \frac{d\sin(lx+\omega t)}{\sin(lx+\omega t)}$$

$$y = -\frac{1}{k} \ln|\sin(lx+\omega t)| + C$$





streamlines: shift these curves in y by C:



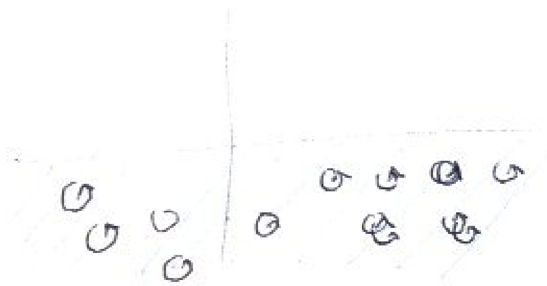
water shaded
streamlines solid lines

Pathlines

$\frac{Dx}{Dt} = \varepsilon \sin(kx_0 + \omega t)$ $\rightarrow x = -\frac{\varepsilon}{\omega}\cos(kx_0 + \omega t) + C$ ↗ call it x_c

$\frac{Dy}{Dt} = -\varepsilon \cos(kx_0 + \omega t)$ $\rightarrow y = -\frac{\varepsilon}{\omega}\sin(kx_0 + \omega t) + D$ ↗ call it y_c

Path lines are circles of radius $\frac{\varepsilon}{\omega}$ around (x_c, y_c)



initial directions of motion vary with x
radius is small

Note: to show circles if it is not obvious:

$$\frac{x - C}{\varepsilon/\omega} = -\cos(kx_0 + \omega t) \qquad \frac{y - D}{\varepsilon/\omega} = -\sin(kx_0 + \omega t)$$

$\cos^2\alpha + \sin^2\alpha = 1$ so

$$\left(\frac{x - C}{\varepsilon/\omega}\right)^2 + \left(\frac{y - D}{\varepsilon/\omega}\right)^2 = 1$$

→ equation of a circle with radius ε/ω and center $x_c = C$, $y_c = D$

For pathline starting from $x = y = t = 0$: $\rightarrow x_0 = 0$ $\quad C = \frac{\varepsilon}{\omega}$ $\quad D = 0$

$$x = -\frac{\varepsilon}{\omega}(\cos(\omega t) - 1)$$

$$y = -\frac{\varepsilon}{\omega}\sin(\omega t)$$

$$\left(\frac{x - \frac{\varepsilon}{\omega}}{\varepsilon/\omega}\right)^2 + \left(\frac{y}{\varepsilon/\omega}\right)^2 = 1$$

Streaklines for generator at $x=0, y=0$

$x = -\frac{\varepsilon}{\omega}\cos(\cancel{kx_0} + \omega t) + C \qquad x(\tau) = 0 \qquad C = +\frac{\varepsilon}{\omega}\cos(\omega\tau)$

ignore

$y = -\frac{\varepsilon}{\omega}\sin(\cancel{kx_0} + \omega t) + D \qquad y(\tau) = 0 \qquad D = +\frac{\varepsilon}{\omega}\sin(\omega\tau)$

$$x = \frac{\varepsilon}{\omega}\left(\cos(\omega\tau) - \cos(\omega t)\right)$$
$$y = \frac{\varepsilon}{\omega}\left(\sin(\omega\tau) - \sin(\omega t)\right)$$

Eliminate τ:

$$\left(\frac{x + \frac{\varepsilon}{\omega}\cos(\omega t)}{\frac{\varepsilon}{\omega}}\right)^2 + \left(\frac{y + \frac{\varepsilon}{\omega}\sin(\omega t)}{\frac{\varepsilon}{\omega}}\right)^2 = 1$$

Streakline at $t=0$ is opposite of path line from $t=0$