Non conservative ~~flows~~ equations

Continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0 \Rightarrow \underbrace{\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i}}_{\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}} + \rho \underbrace{\frac{\partial u_i}{\partial x_i}}_{\operatorname{div}\vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}}$$

$$\boxed{\frac{1}{\rho}\frac{D\rho}{Dt} + \operatorname{div}\vec{v} = 0}$$

Non conservative continuity

Incompressible flow: $\boxed{\frac{D\rho}{Dt} = 0 \Rightarrow \operatorname{div}\vec{v} = 0}$ incompressible

ρ does not have to be constant for $\operatorname{div}\vec{v} = 0$.

Only $\frac{D\rho}{Dt} = 0$ must be zero: density of particles unchanged with time

e.g. – 2 incompressible ~~unmixable~~ inmiscible liquids of different density

– water of varying salinity, ignoring diffusion of salt

Momentum

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_j u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ji}}{\partial x_j} + \rho g_i$$

$$u_i \frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho u_j}{\partial x_j} + \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j}$$

$$\underbrace{u_i\left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j}\right)}_{\to 0 \text{ continuity}} + \underbrace{\rho\underbrace{\left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right]}_{\frac{Du_i}{Dt} = a_i}}_{\rho a_i} = \underbrace{-\frac{\partial p}{\partial x_i} + \mu \frac{\partial \tau_{ji}}{\partial x_j} + \rho g_i}_{\text{force on fluid particles per unit volume}}$$

Pressure force on a fluid particle per unit volume	$\left\{-\frac{\partial p}{\partial x_i}\right\} = -\nabla p$
Viscous force " " " " " "	$\left\{+\frac{\partial \tau_{ji}}{\partial x_j}\right\} = \nabla \overline{\overline{\tau}}$
Gravity force on " " " " "	$\{\rho g_i\} = \rho \vec{g}$

Navier-Stokes equation written out

Continuity: $\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right]$

Momentum: $\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x$

Stress tensor $\tau_{xx}^{tot} = -p + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial u}{\partial x}$

$\tau_{xy}^{tot=visc} = \mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ $\quad \underset{\text{Stokes}}{\lambda} = -\frac{2}{3}\mu$

Note : ~~while~~ Say $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$ is simply $\frac{Du}{Dt}$,
but you cannot evaluate ~~this~~ $\frac{Du}{Dt}$ directly given $\vec{v} = \vec{v}(\vec{r}, t)$,
you must use Eulerian partial derivatives. $\frac{Du}{Dt}$ is
just for physical understanding.

Still need our fifth equation for the 5 unknowns ρ, u_1, u_2, u_3, p

Hydrostatics

$u = v = w = 0$ $\qquad$ viscous ~~shear~~ stresses zero for Newtonian

continuity $\frac{\partial \rho}{\partial t} = 0$

momentum $\cancel{-\frac{\partial p}{\partial x_i} + \rho g_i = 0}$ $\qquad \rho\vec{g} = \cancel{-\rho\nabla gh} - \rho g\nabla h$

$-\nabla p + \rho\vec{g} = 0$

$\boxed{\nabla p + \rho g \nabla h = 0}$ → can show that ρ and p only can depend on h *

$\boxed{\text{Incompressible, constant density and } g\text{: } p = p_{HS} = C - \rho g h}$

Kinetic Pressure

Use for ~~con~~ constant density flows without free surfaces:

$\cancel{p = p_{kin} - \rho g h}$ $\boxed{p_{kin} = p + \rho g h}$ → p_{HS}

Then $-\frac{\partial p}{\partial x} + \rho g_x = -\frac{\partial p_{kin}}{\partial x}$ $\qquad$: no longer a gravity g in the equations!

~~But~~ Good for homework