Energy:

Like for momentum, energy can be differentiated ~~and~~ out and continuity applied to give:

$$-\frac{\partial p}{\partial x_i}u_i - p\frac{\partial u_i}{\partial x_j} \qquad u_i\frac{\partial \tau_{ji}}{\partial x_j} + \tau_{ji}\frac{\partial u_i}{\partial x_j}$$

$$\rho\left[\frac{\partial(e+\frac{1}{2}u_iu_i)}{\partial t} + u_j\frac{\partial(e+\frac{1}{2}u_iu_i)}{\partial x_j}\right] = -\frac{\partial p u_i}{\partial x_i} + \frac{\partial \tau_{ji}u_i}{\partial x_j} + \rho g_i u_i - \frac{\partial q_i}{\partial x_i}$$

heat

Use "Mechanical Energy" to get rid of the kinetic energy terms:

$$u_i\Big\{\underbrace{\rho\left[\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right] = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ji}}{\partial x_j} + \rho g_i}_{\text{momentum equation}}\Big\}$$

note $u_i du_i = d\frac{1}{2}u_iu_i$

$\cancel{u_i d\tau_{ji} = d(\tau_{ji}u_i) - \tau_{ji}du_i}$

$$\boxed{\rho\left[\frac{\partial \frac{1}{2}u_iu_i}{\partial t} + u_j\frac{\partial \frac{1}{2}u_iu_i}{\partial x_j}\right] = -u_i\frac{\partial p}{\partial x_i} + u_i\frac{\partial \tau_{ji}}{\partial x_j} + \rho g_i u_i}$$

"Mechanical" energy equation (a consequence of momentum, not a fundamental/new equation)

Subtract from energy equation → kinetic energy terms drop out:

$\cancel{\rho\left[\frac{\partial e}{\partial t} + u_j\frac{\partial e}{\partial x_j}\right] = +p\frac{D\rho}{Dt} + \varepsilon +}$

$$\boxed{\rho\left[\frac{\partial e}{\partial t} + u_j\frac{\partial e}{\partial x_j}\right] = -p\frac{\partial u_i}{\partial x_i} + \tau_{ji}\frac{\partial u_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}}$$

Thermal energy equation: alternative to the original one

Interpretation:

$$\frac{\partial e}{\partial t} + u_j\frac{\partial e}{\partial x_j} = \frac{De}{Dt}$$

$$-p\frac{\partial u_i}{\partial x_i} = -p\,\mathrm{div}\,\vec{v} \overset{\text{continuity}}{=} +\frac{p}{\rho}\frac{D\rho}{Dt} = \cancel{t\rho p \frac{Dv}{Dt}}$$

specific volume $\frac{1}{\rho}$

s entropy

Thermo: $\rho\left[De - \frac{p}{\rho^2}D\rho\right] = \rho T Ds$

$De \to$ "du", $\; + p$ "dv"

$$\rho\frac{TDs}{Dt} = \tau_{ji}\frac{\partial u_i}{\partial x_j} + \frac{\partial q_i}{\partial x_i}$$

Interpretation

$$\underbrace{\rho\left[\frac{\partial e}{\partial t}+u_j\frac{\partial e}{\partial x_j}\right]}_{\frac{De}{Dt}} = \underbrace{-p\frac{\partial u_i}{\partial x_i}}_{\text{continuity}\ +\frac{p}{\rho}\frac{D\rho}{Dt}} + \tau^v_{ji}\frac{\partial u_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}$$

$$\frac{1}{2}\left(\tau^v_{ji}\frac{\partial u_i}{\partial x_j}+\tau^v_{ij}\frac{\partial u_j}{\partial x_i}\right)$$

$$\tau^v_{ij}\,\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$$

(if $\overline{\overline{\tau}}$ is symmetric)

$$\tau_{ij}\,s_{ij}$$

$$\rho\left[\frac{De}{Dt}-\frac{p}{\rho^2}\frac{D\rho}{Dt}\right]=$$

$$\rho\left[\frac{De}{Dt}+p\frac{D\frac{1}{\rho}}{Dt}\right]=$$

$$\underbrace{\text{"}du\text{"}+p\,\text{"}dv\text{"}}_{\text{Thermo: } Tds \text{ (Gibbs)}}$$

$\tau : \mathbb{S} \equiv \tau_{ij}s_{ij}$ is called the dissipation ε (Kundu $\rho\varepsilon$)

$$\rho T\frac{Ds}{Dt} = \qquad \boldsymbol{\varepsilon} - \frac{\partial q_i}{\partial x_i}$$

$$\boxed{\varepsilon \equiv \tau_{ij}\,s_{ji} \equiv \overline{\overline{\tau}}:\mathbb{S} \text{ is dissipation per unit volume} \quad \to \text{Kundu: } \rho\boldsymbol{\varepsilon}}$$

$$\rho\frac{Ds}{Dt} = \frac{\varepsilon}{T} - \frac{1}{T}\frac{\partial q_i}{\partial x_i} = \frac{\varepsilon}{T} - \frac{\partial q_i/T}{\partial x_i} + q_i\frac{\partial 1/T}{\partial x_i}$$

$$\boxed{\rho\frac{Ds}{Dt} = -\frac{\partial q_i/T}{\partial x_i} + \frac{\varepsilon}{T} - \frac{q_i}{T^2}\frac{\partial T}{\partial x_i}}$$

Entropy equation: ~~equivalent~~ alternative to original Energy equation

Take an insulated ~~region~~ Lagrangian region and integrate over it

$$\int_{\forall_{MR}}\frac{Ds}{Dt}\,\overset{\rho\,d\forall}{dm} = \int_{A_{MR}} \cancel{-\frac{q_i n_i}{T}dA}^{\text{insulated}} + \int_{\forall_{MR}}\left(\frac{\boldsymbol{\varepsilon}}{T}-\frac{q_i}{T^2}\frac{\partial T}{\partial x_i}\right)d\forall$$

$$\frac{D\,S_{MR}}{Dt} = \int_{\forall_{MR}}\left(\frac{\varepsilon}{T}-\frac{q_i}{T^2}\frac{\partial T}{\partial x_i}\right)d\forall$$

Second law: The integral in the right hand side better be positive!!

If Fourier law $q_i = -k \frac{\partial T}{\partial x_i}$ then $-\frac{q_i}{T^2}\frac{\partial T}{\partial x_i} = +k\frac{1}{T^2}\left(\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_i}\right)$

is indeed positive always (if $k > 0$).

$\boxed{k \text{ must be positive}}$

ε??

$$\varepsilon = \tau_{ij} s_{ij} = s_{ij}\left(\lambda \delta_{ij}\frac{\partial u_k}{\partial x_k} + 2\mu s_{ij}\right)$$

$$= 2\mu s_{ij}s_{ij} + \lambda s_{ii}s_{kk}$$

Go to principle axes:

$$s_{ij}s_{ij} = s_1^2 + s_2^2 + s_3^2 = |\vec{s}|^2 \rightarrow \text{if } \vec{s} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$$

$$s_{ii}s_{kk} = (s_1 + s_2 + s_3)^2 = (\vec{s}\cdot\vec{1})^2 \rightarrow \text{if } \vec{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= |\vec{s}|^2 \underbrace{|\vec{1}|^2}_{3} \cos^2\vartheta$$

$$\varepsilon = |\vec{s}|^2\{2\mu + 3\lambda\cos^2\vartheta\}$$

If we demand that ε is positive at every point, and it must be: we can always use a uniform straining in principal axes as flowfield

$\boxed{\mu \text{ must be positive}}$ (case $\cos\vartheta = 0$) if compressible

Also $\boxed{3\lambda \geq -2\mu}$ required (cases $\cos\vartheta = \pm 1$)

Stokes hypothesis: $\lambda = -\frac{2}{3}\mu$ allows ε to be zero in the case of uniform expansion ($\cos\vartheta = 0$ so $s_1 = s_2 = s_3$)

(I find it hard to believe. If you compress air rapidly, you create shockwaves. If you slow down that process, these shock waves thicken but are still dissipating heat. Why would that not be described by a Newtonian fluid viscosity?)

Note: Stokes hypothesis might apply to monatomic gasses, depending on who you talk to.

Note: experimental data might be outside the range of where the Newtonian assumption applies