Interpretation

$$\underbrace{\underbrace{\rho\left[\frac{\partial e}{\partial t} + u_j \frac{\partial e}{\partial x_j}\right]}_{\frac{De}{Dt}} = \underbrace{-p\frac{\partial u_i}{\partial x_i}}_{\text{continuity } +\frac{p}{\rho}\frac{D\rho}{Dt}}} + \tau_{ji}^{v}\frac{\partial u_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}$$

$$\underbrace{\frac{1}{2}\left(\tau_{ji}^{v}\frac{\partial u_i}{\partial x_j} + \tau_{ij}^{v}\frac{\partial u_j}{\partial x_i}\right) = \tau_{ij}^{v}\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{(if } \bar{\bar{\tau}} \text{ is symmetric)}}$$

$$\rho\left[\frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt}\right] = \tau_{ij} S_{ij}$$

$$\rho\left[\frac{De}{Dt} + p\frac{D\frac{1}{\rho}}{Dt}\right] =$$

$\tau : S \equiv \tau_{ij} S_{ij}$ is called the dissipation ε (Kundu $\rho\varepsilon$)

$$\underbrace{\text{"}du\text{"} + p\,\text{"}dv\text{"}}_{\text{Thermo: } Tds \text{ (Gibbs)}}$$

$$\rho T\frac{Ds}{Dt} = \varepsilon - \frac{\partial q_i}{\partial x_i}$$

$\boxed{\varepsilon \equiv \tau_{ij} S_{ji} \equiv \bar{\bar{\tau}} : S \text{ is dissipation per unit volume} \quad \to \text{Kundu: } \rho\varepsilon}$

$$\rho\frac{Ds}{Dt} = \frac{\varepsilon}{T} - \frac{1}{T}\frac{\partial q_i}{\partial x_i} = \frac{\varepsilon}{T} - \frac{\partial q_i/T}{\partial x_i} + q_i\frac{\partial 1/T}{\partial x_i}$$

$$\boxed{\rho\frac{Ds}{Dt} = -\frac{\partial q_i/T}{\partial x_i} + \frac{\varepsilon}{T} - \frac{q_i}{T^2}\frac{\partial T}{\partial x_i}}$$

Entropy equation: ~~equivalent~~ alternative to original Energy equation

Take an insulated ~~region~~ Lagrangian region and integrate over it

$$\int_{V_{MR}} \frac{Ds}{Dt}\, \overset{\rho\, dV}{dm} = \int_{A_{MR}} \cancel{-\frac{q_i n_i}{T}\, dA}^{\text{insulated}} + \int_{V_{MR}}\left(\frac{\varepsilon}{T} - \frac{q_i}{T^2}\frac{\partial T}{\partial x_i}\right) dV$$

$$\frac{D\, S_{MR}}{Dt} = \int_{V_{MR}}\left(\frac{\varepsilon}{T} - \frac{q_i}{T^2}\frac{\partial T}{\partial x_{i}}\right) dV$$

Second law: The integral in the right hand side better be positive !!

If Fourier law $q_i = -k \frac{\partial T}{\partial x_i}$ then $-\frac{q_i}{T^2} \frac{\partial T}{\partial x_i} = +k \frac{1}{T^2} \left(\frac{\partial T}{\partial x_i} \frac{\partial T}{\partial x_i} \right)$

is indeed positive always (if $k > 0$).

$\boxed{k \text{ must be positive}}$

ε ??

$$\varepsilon = \tau_{ij} \, s_{ij} = s_{ij} \left(\lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + 2\mu \, s_{ij} \right)$$

$$= 2\mu \, s_{ij} s_{ij} + \lambda \, s_{ii} \, s_{kk}$$

Go to principle axes : $s_{ij} s_{ij} = s_1^2 + s_2^2 + s_3^2 = |\vec{s}|^2$ → if $\vec{s} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$

$s_{ii} \, s_{kk} = (s_1 + s_2 + s_3)^2 = (\vec{s} \cdot \vec{1})^2$ → if $\vec{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$= |\vec{s}|^2 \, \underbrace{|\vec{1}|^2}_{3} \, \text{“}\cos^2 \vartheta\text{”}$

$$\varepsilon = |\vec{s}|^2 \left\{ 2\mu + 3\lambda \cos^2 \vartheta \right\}$$

If we demand that ε is positive at every point, → and it must be: we can always use a uniform straining in principal axes as flowfield

$\boxed{\mu \text{ must be positive}}$ (case $\cos\vartheta = 0$) → if compressible

Also $\boxed{3\lambda \geq -2\mu}$ ~~are~~ required (cases $\cos\vartheta = \pm 1$)

Stokes hypothesis: $\lambda = -\frac{2}{3}\mu$ allows ε to be zero in the case of uniform expansion ($\cos\vartheta = 0$ so $s_1 = s_2 = s_3$)

(I find it hard to believe. If you compress air rapidly, you create shockwaves. If you slow down that process, these shock waves thicken but are still dissipating heat. Why would that ~~not~~ be described by a viscosity ?)

Newtonian fluid ~~[illegible]~~

Note: Stokes hypothesis might apply to monatomic gasses, depending on who you talk to.

Note: experimental data might be outside the range of where the Newtonian assumption applies

~~[illegible]~~

~~[illegible]~~

Euler equations

"Inviscid" flow: assume $\mu = 0$ (so $\lambda = 0$) and $k = 0$

energy becomes $\dfrac{Ds}{Dt} = 0 \implies \dfrac{Dp}{Dt} = \left(\dfrac{\partial p}{\partial \rho}\right)_s \dfrac{D\rho}{Dt}$

Definition: $\boxed{a = \sqrt{\left(\dfrac{\partial p}{\partial \rho}\right)_{s\ \text{constant}}}\ : \text{"speed of sound"}}$

For ideal gasses with constant c_v:
($c_p = c_v + R$, $\gamma = \dfrac{c_p}{c_v}$ constant too).

$$a = \sqrt{\gamma \frac{p}{\rho}}$$

Euler equations:

often omitted for gases

$$\boxed{\begin{aligned} &\frac{1}{\rho}\frac{D\rho}{Dt} + \operatorname{div}\vec{v} = 0 \\ &\rho \frac{D\vec{v}}{Dt} = -\nabla p + \rho \vec{g} \\ &\frac{Dp}{Dt} = a^2 \frac{D\rho}{Dt} \end{aligned}}$$

"Incompressible" Navier-Stokes (Euler) equations

[Assume ρ and μ are constants] (and k)

Viscous stress force is then

$$\frac{\partial \tau_{ji}}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\lambda \cancelto{0}{\operatorname{div}\vec{v}} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]$$

$$= \mu \frac{\partial^2 u_i}{\partial x_j^2} + \mu \frac{\partial}{\partial x_i}\left(\cancel{\frac{\partial u_j}{\partial x_j}}\right) \quad = \operatorname{div}\vec{v} = 0$$

So:

Viscous force/unit volume for "incompressible"

$$\nabla\cdot\bar{\bar{\tau}} = \mu\nabla^2\vec{u} = \mu\left[\frac{\partial^2\vec{v}}{\partial x^2} + \frac{\partial^2\vec{v}}{\partial y^2} + \frac{\partial^2\vec{v}}{\partial z^2}\right]$$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the "Laplacian"

In non-Cartesian coordinates, look it up

"Incompressible" Navier-Stokes equations:

$$\rho\left(\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla p + \mu\nabla^2\vec{v} + \rho\vec{g}$$

E.g. x-component:

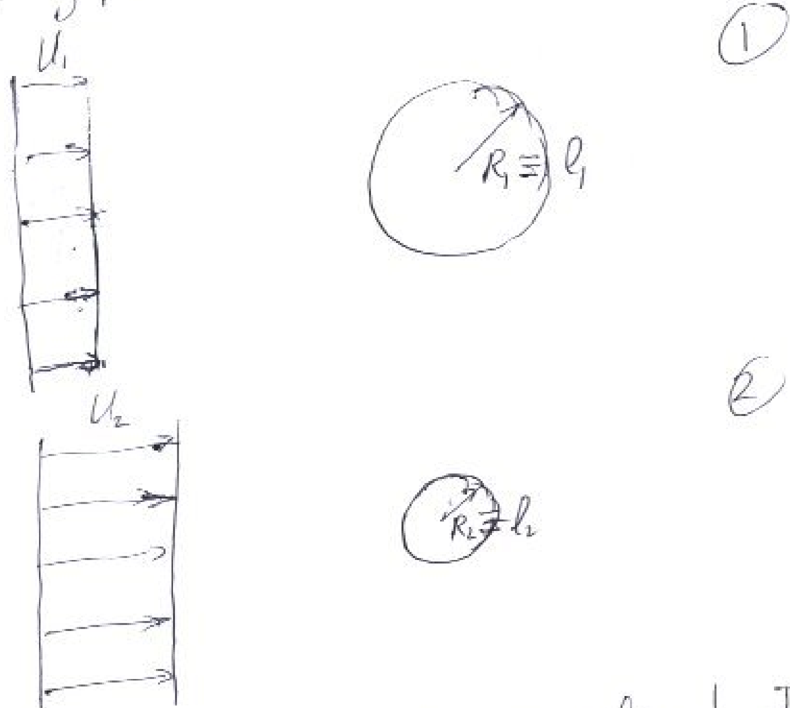
$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho g_x$$

(Remember: Normally we can rid of $\rho\vec{g}$ using the kinetic pressure)

$$\rightarrow p_{kin} = p + \rho g h$$

Similarity and nondimensionalization (Kundu 4.11, Panton?)

Suppose we have two ~~cylinders, each in~~ geometrically similar bodies (e.g. both circular cylinders) both sitting in a steady uniform stream coming from the left. Are the flow fields similar??



are the flow fields similar?? I.e. if I scale the size of flow field 1 down by a factor $\frac{R_2}{R_1}$, do I get flow field (2)?? (Maybe also scale down time?)

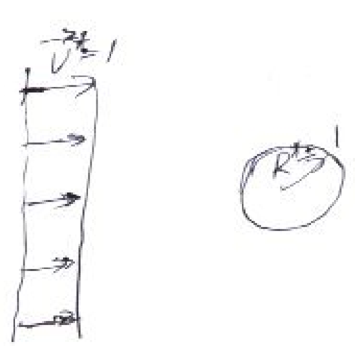
To figure out, nondimensionalize! For generic flow around a circular cylinder, define a characteristic length l. Take $l = R$, please, not D like experimentalists do. Define a characteristic velocity, U normally the velocity far from the body.

Now define nondimensional variables

$$\vec{x} = l\,\vec{x}^* \qquad t = \frac{l}{U}t^* \qquad \vec{v} = U\vec{v}^* \qquad p = p_\infty + \rho U^2 p^*$$

then in both flows (1) and (2) the cylinder has nondimensional radius $r^* = 1$

then in both flows, the nondimensional velocity at infinity is $\vec{v}^* = 1$



Plug it into the ~~equations~~ Navier-Stokes equations

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right) = -\nabla p + \mu \nabla^2 \vec{v}$$

$$\rho \frac{\partial \vec{v}^*}{\partial t^*}\frac{UU}{l} + \rho\, \vec{v}^*\cdot\nabla^*\vec{v}^*\frac{U^2}{l} = -\nabla^* p^* \frac{\rho U^2}{l} + \mu \nabla^{*2}\vec{v}^*\frac{U}{l^2} \qquad \frac{l}{\rho U^2}$$

$$\boxed{\frac{\partial \vec{v}^*}{\partial t^*} + \vec{v}^*\cdot\nabla^*\vec{v}^* = -\nabla^* p^* + \frac{\mu}{\rho U l}\nabla^{*2}\vec{v}^*}$$

Navier-Stokes

Non-Dimensional incompressible

By definition $\frac{1}{Re}$, $\boxed{Re \equiv \frac{\rho U l}{\mu}}$

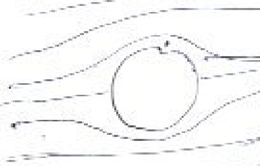
Re is called the "Reynolds number"

Conclusion: if flows ① and ② have different Reynolds numbers,

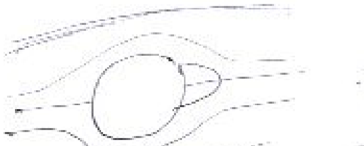
$$\frac{\rho_1 U_1 l_1}{\mu_1} \neq \frac{\rho_2 U_2 l_2}{\mu_2}$$

They will not be similar

~~E.g. very low~~ Re:

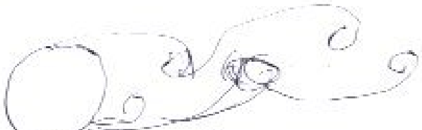


very low Re



quite low Re

quite low Re



somewhat low

flow became unsteady

higher,

very much turbulent

See Panton 325-326 !

Dependent variables : ~~how about~~