Buckingham Π theorem

Method:

1) Identify all independent variables that determine the flow field:

   Example: cylinder in incompressible flow

   $$\underset{\substack{\uparrow \\ \text{dep var}}}{D} = f(\rho, l, U, \mu, g, \Omega, A, \ldots)$$

2) Select a minimal number of variables to make all other variables nondimensional with. (= no units)
   Usually three variables to get rid of length, time, and mass units. (If some independent variables include temperature, be sure to include a suitable other variable like R in the list of independent variables)

   Example

   Select $l : L$ (length), $U : \frac{L}{T}$ (T = time), $\rho : \frac{M}{L^3}$ (M → mass) → Test that no nondimensional parameter can be formed with the selected parameters

3) Make all other variables except the selected ones ~~independent~~ nondimensional using the selected ones

   Example

   $$\mu = \frac{\text{Force}}{\text{Area}} \frac{\text{length}}{\text{velocity}} = \frac{M \frac{L}{T^2} L}{L^2 (L/T)} = \frac{M}{LT}$$

   $$\Pi_\mu = \mu l^a U^b \rho^c \rightarrow M L^{-1} T^{-1} L^a L^b T^{-b} M^c L^{-3c}$$

   Now the net powers of L, T, M must be zero

   M power: $1 + c = 0 \rightarrow c = -1$

   T power: $-1 - b = 0 \rightarrow b = -1$

   L power: $-1 + a + b - 3c = 0 \rightarrow -1 + a - 1 + 3 = 0 \quad a = -1$

   $$\Longrightarrow \Pi_\mu = \frac{\mu}{\rho U l}$$

(Note: if you do not get unique values for a, b, c, your selection of variables in 2 was incorrect. Or you made a mistake.)

$g \rightarrow \frac{L}{T^2} \quad \Pi_g = g \, l^a U^b \rho^c \longrightarrow L^1 T^{-2} L^a L^b T^{-b} M^c L^{-3c}$

M power: $c = 0$
T power: $b = -2$
L power $1 + a - 2 = 0 \quad a = 1$

$$\Pi_g = \frac{g l}{U^2}$$

$\Omega \rightarrow \frac{1}{T}$ by inspection $\quad \Pi_\Omega = \frac{\Omega l}{U}$

$A \rightarrow L$ " " $\quad \Pi_A = \frac{A}{l}$

Dependent variable of interest

$D \rightarrow M \frac{L}{T^2} \quad D l^a U^b \rho^c \longrightarrow M^1 L^1 T^{-2} L^a L^b T^{-b} M^c L^{-3c}$

M : $c = -1$
T : $b = -2$
L : $1 + a - 2 + 3 = 0 \quad a = -2$

$$\Pi_D = \frac{D}{\rho U^2 l^2}$$

Now $\Pi_D = f^*(\rho, l, U, \Pi_\mu, \Pi_g, \Pi_\Omega, \Pi_A, \ldots)$

Buckingham Π-theorem: f^* cannot vary with ρ, l, and U. Otherwise, if I change from say meter to inch units for length, the physics would change.

$$\Pi_D = f^*(\Pi_\mu, \Pi_g, \Pi_\Omega, \Pi_A, \ldots)$$

(Note: If you are unsure whether your selected variables are minimal, make sure that you cannot create a nondimensional combination with them

$\Pi_{sel} = l^a U^b \rho^c \longrightarrow L^a L^b T^{-b} M^c L^{-3c}$

$M \rightarrow c = 0 \quad T \rightarrow b = 0 \quad L \rightarrow a = 0$ O.K.

Alternately, test that the created Π groups are unique. If you do not find unique values for a, b, c in the Π groups, the selected variables are dependent,

Stokes flow   Kundu 8.6, Panton 21.1 – 21.8

Assumes very low Reynolds number $\frac{\rho U l}{\mu}$

So very high viscosity (relative to conditions)

(Or very small body or very slow flow)

Scaled Navier-Stokes equations:

$$\nabla^* \cdot \vec{v}^* = 0 \qquad \text{nothing happens if } Re \to 0$$

$$\frac{\partial v^*}{\partial t^*} + \vec{v}^* \cdot \nabla^* \vec{v}^* = -\nabla p^* + \underbrace{\frac{1}{Re}}_{\text{blows up}} \nabla^{*2} \vec{v}^*$$

$-\nabla p^*$: p^* can respond by getting big too, proportional to the Reynolds number

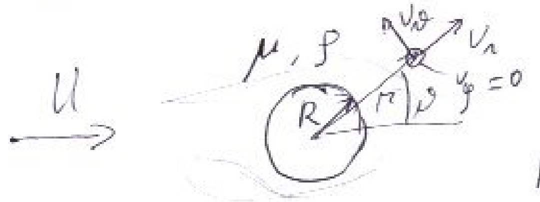
$\vec{v}^* \cdot \nabla^* \vec{v}^*$: v^* cannot become big.

$\frac{\partial v^*}{\partial t^*}$: evolution can become very fast (small time scale)

Conclusion:

$$\boxed{\begin{array}{l}\text{In Stokes flow approximation,} \\ \text{the } \vec{v}^* \cdot \nabla^* \vec{v}^* \text{ terms drop out} \\ \hline \text{(Only for very low } Re\text{, and} \\ \text{even then, be careful)}\end{array}}$$

Stokes flow around a sphere (Incompressible, highly viscous)



At large r, $\vec{v} \sim U\hat{i}$ → $v_r \sim U\cos\theta$, $v_\theta \sim -U\sin\theta$

Maybe $v_r = f(r)\cos\theta$? $v_\theta = g(r)\sin\theta$?

Appendix B:

$\nabla\cdot\vec{v} = 0$

$$\frac{\partial v_r}{\partial r} + \frac{2}{r}v_r + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\cos\theta}{r\sin\theta}v_\theta = 0$$

$$f'\cancel{\cos\theta} + \frac{2}{r}f\cancel{\cos\theta} + \frac{g}{r}\cancel{\cos\theta} + \frac{g}{r}\cancel{\cos\theta} = 0$$

$$g = -\frac{1}{2}\left[rf' + 2f\right] = -\frac{1}{2}rf' - f \quad ✓$$

Looks good!

$$\rho\Big[\underbrace{\cancel{\frac{\partial \vec{v}}{\partial t}}}_{\text{steady}} + \underbrace{\cancel{\vec{v}\cdot\nabla\vec{v}}}_{\text{Stokes approx}}\Big] = -\frac{1}{\rho}\nabla p + \mu\nabla^2\vec{v}$$

Appendix B: ($\frac{\partial}{\partial\varphi} = 0$, $v_\varphi = 0$)

$$0 = -\frac{\partial p}{\partial r} + \mu\Big[\frac{\partial^2 v_r}{\partial r^2} + \frac{2}{r}\frac{\partial v_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial v_r}{\partial\theta} - \frac{2}{r^2}v_r - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{2\cos\theta}{r^2\sin\theta}v_\theta\Big]$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial\theta} + \mu\Big[\frac{\partial^2 v_\theta}{\partial r^2} + \frac{2}{r}\frac{\partial v_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial\theta} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r^2\sin^2\theta}\Big]$$

$$0 = -\frac{\partial p}{\partial r} + \mu\Big[f''\cos\theta + \frac{2}{r}f'\cos\theta - \frac{1}{r^2}f\cos\theta - \frac{1}{r^2}f\cos\theta - \frac{2}{r^2}f\cos\theta - \frac{2}{r^2}g\cos\theta - \frac{2}{r^2}g\cos\theta\Big]$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial\theta} + \mu\Big[g''\sin\theta + \frac{2}{r}g'\sin\theta - \frac{1}{r^2}g\sin\theta + \frac{1}{r^2}g\frac{\cos^2\theta}{\sin\theta} - \frac{2}{r^2}f\sin\theta - \frac{1}{r^2}g\frac{1}{\sin\theta}\Big]$$

combine → $-\frac{1}{r^2}g\sin\theta$

$$0 = -\frac{\partial p}{\partial r} + \mu\left[f'' + \frac{2}{r}f' - \frac{4}{r^2}f - \frac{4}{r^2}g\right]\cos\theta$$ 40

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[g'' + \frac{2}{r}g' - \frac{2}{r^2}g - \frac{2}{r^2}f\right]\sin\theta$$

Try $p = \cancel{\mu}h(r)\cos\theta$

$$-h' + f'' + \frac{2}{r}f' - \frac{4}{r^2}f - \frac{4}{r^2}g = 0 \qquad \cancel{g = -\tfrac{1}{2}rf' - f}$$

$$\frac{1}{r}h + g'' + \frac{2}{r}g' - \frac{2}{r^2}g - \frac{2}{r^2}f = 0$$

Now we want to create one ODE in one unknown (f) by eliminating g and h

To eliminate h:

$$-h' + f'' + \frac{2}{r}f' - \frac{4}{r^2}f - \frac{4}{r^2}g = 0$$

$$h + rg'' + 2g' - \frac{2}{r}g - \frac{2}{r}f = 0 \Longrightarrow \frac{\partial}{\partial r}$$

$$h' + rg''' + 3g'' - \frac{2}{r}g' + \frac{2}{r^2}g - \frac{2}{r}f' + \frac{2}{r^2}f \quad +$$

$$0 = f'' + \frac{2}{r}f' - \frac{4}{r^2}f - \frac{4}{r^2}g + rg''' + 3g'' - \frac{2}{r}g' - \frac{2}{r^2}g$$

$$0 = f'' - \frac{2}{r^2}f + rg''' + 3g''$$

To eliminate g, we continue

$$g = -\frac{1}{2}rf' - f \qquad g' = -\frac{1}{2}rf'' - \frac{3}{2}f'$$

$$g'' = -\frac{1}{2}rf''' - 2f'' \qquad g''' = -\frac{1}{2}rf'''' - \frac{5}{2}f''' \cancel{-6f''}$$

$$-\frac{3}{2}rf''' + f'' + \frac{3}{r}f'$$

$$0 = f'' - \frac{2}{r^2}f - \frac{1}{2}r^2f'''' - \frac{5}{2}rf''' - \frac{3}{2}rf''' + f'' + \frac{1}{r}f' + \frac{2}{r^2}f = 0 \quad * -2r^2$$

$$r^4f'''' + 8r^3f''' + 8r^2f'' - 8rf' = 0$$

"Euler" equation, normally has solutions that are powers of r

If $f = r^n$, then

$$n(n-1)(n-2)(n-3) + 8n(n-1)(n-2) + 8n(n-1) - 8n = 0$$

$\cancel{n^4 - 5n^3 + 6n^2 - n^2 - 5n + 6}$

One real: $n = 0$

other three roots:

$$(n-1)(n-2)(n-3) + 8(n-1)(n-2) + \underbrace{8(n-1) - 8}_{8(n-2)} = 0$$

common factor $(n-2)$ → root $n=2$

other two roots

$$(n-1)(n-3) + 8(n-1) + 8 = 0$$

$$n^2 - 4n + 3 + 8n = 0 \qquad n^2+4n+3 = 0 = (n+1)(n+3)$$

roots $n=-1$ $n=-3$ → blows up at ∞, but velocity should not, so A_3 must be 0

Total solution: combination of the four powers

$$\to f = A_0 + \frac{A_1}{r} + \frac{A_2}{r^3} + \cancel{A_3 r^2}$$

$$g = -f - \frac{1}{2} r f' = -A_0 + \frac{1}{2}\frac{A_1}{r} - \frac{A_1}{r} - \frac{A_2}{r^3} + \frac{3}{2}\frac{A_2}{r^3}$$

$$= -A_0 - \frac{A_1}{2r} + \frac{A_2}{2r^3}$$

Apply B.C. to find A_0, A_1, A_2

For $r \to \infty$ $f \sim U \to A_0 = U$ then $g \sim -U$ as it should

At $r = R$ $f = 0$ $U + \frac{A_1}{R} + \frac{A_2}{R^3} = 0$

$$g = 0 \quad -U - \frac{A_1}{2R} + \frac{A_2}{2R^3} = 0$$

$$\left(\begin{array}{cc|c} \frac{1}{R} & \frac{1}{R^3} & -U \\ -\frac{1}{2R} & \frac{1}{2R^3} & U \end{array}\right) \supset \frac{1}{2} \quad \left(\begin{array}{cc|c} \frac{1}{R} & \frac{1}{R^3} & -U \\ 0 & \frac{1}{R^3} & \frac{1}{2}U \end{array}\right) \to A_2 = \frac{UR^3}{2} \Big\} \; A_1 = -\frac{3}{2}UR$$

$$\underline{f = U\left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right)} \; ; \quad g = U\left(-1 + \frac{3R}{4r} + \frac{R^3}{4r^3}\right)$$

$$h = -r g'' - 2g' + \frac{2}{r} \underset{-f - \frac{1}{2} r f'}{g} + \frac{2}{r} f = -r g'' - 2g' - f'$$

$$= U\left(-\frac{3R}{2r^2} - \cancel{\frac{3R^3}{r^4}} + \cancel{\frac{3R}{2r^2}} + \cancel{\frac{3R^3}{2r^4}} - \cancel{\frac{3R}{2r^2}} + \cancel{\frac{3R^3}{2r^4}}\right)$$

$$= U\left(-\frac{3R}{2r^2}\right)$$

$$\underline{h = -\frac{3UR}{2r^2}}$$

v_r, v_θ, p follow.