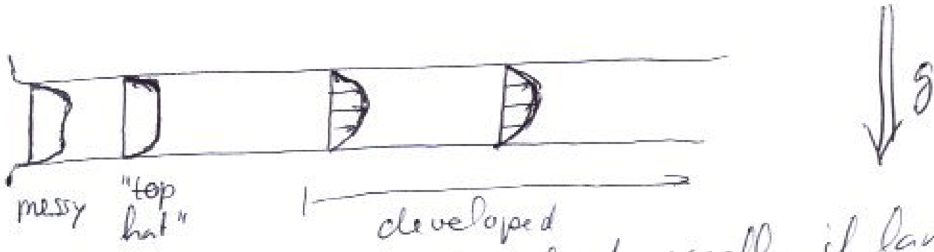
Couette & Plane Poiseuille Flow



The developed region can be solved exactly if laminar!

Assumptions
1. Incompressible
2. Newtonian
3. Laminar, steady
4. 2D $w = 0$, $\frac{\partial}{\partial z} = 0$
5. velocity is independent of x

Method: (Start with easiest equation) (solve as far as possible)

1) Plug assumptions into continuity,

$$\cancelto{(1)}{\frac{1}{\rho}\frac{D\rho}{Dt}} + \cancelto{(5)}{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y} + \cancelto{(4)}{\frac{\partial w}{\partial z}} = 0$$

$$\frac{\partial v}{\partial y} = 0 \longrightarrow v = v(x, \cancelto{(4)}{z}, \cancelto{(3)}{t})$$

(Use wall Boundary condition) $v = 0$ at $y = 0$, any x

$$v = v(x) = 0 \text{ any } x \quad \boxed{v = 0} \quad (6)$$

2) Plug assumptions and current results into (y) momentum

$$\underbrace{\rho \cancelto{(6)}{\frac{Dv}{Dt}}}_{\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)} = -\frac{\partial p}{\partial y} + \mu \cancelto{(6)}{\nabla^2 v} - \rho g$$

integration constant!

$$0 = \frac{\partial p}{\partial y} + \rho g \quad \boxed{\text{Phil} =} \boxed{p + \rho g y = p_0(x, \cancelto{(4)}{z}, \cancelto{(3)}{t})} \quad (7)$$

3) Plug assumptions and current results into x-momentum

$$\rho\left(\cancelto{(3)}{\frac{\partial u}{\partial t}} + \cancelto{(5)}{u\frac{\partial u}{\partial x}} + \cancelto{(6)}{v\frac{\partial u}{\partial y}} + \cancelto{(4)}{w\frac{\partial u}{\partial z}}\right) = -\frac{\partial p}{\partial x} + \mu\left(\cancelto{(5)}{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \cancelto{(4)}{\frac{\partial^2 u}{\partial z^2}}\right)$$

$$0 = -p_0'(x) + \mu \frac{\partial^2 u}{\partial y^2}$$

4) Examine restrictions on the $\hat{p}_0(x)$ pressure integration constant: (49)

$\hat{p}_0(x) = \mu \frac{\partial^2 u}{\partial x^2}$ is independent of x according to (5)

$\rightarrow \hat{p}_0$ is a constant.

$\rightarrow p_0(x) = \frac{dp_0}{dx} x + p_1$

$\rightarrow \boxed{p + \rho g y = \frac{dp_0}{dx} x + p_1}$ (7) $\boxed{\frac{dp_0}{dx} \text{ constant}}$ (5) (4) (3)

$u = u(\cancel{x}, y, \cancel{z}, \cancel{t})$

5) Find u from $\cancel{x}$-momentum and B.C.

$\mu \frac{\partial^2 u}{\partial y^2} = \mu \frac{d^2 u}{dy^2} = \frac{dp_0}{dx}$ $\rightarrow \mu \frac{du}{dy} = \frac{dp_0}{dx} y + D \rightarrow \mu u = \frac{dp_0}{dx} \frac{1}{2} y^2 + Dy + E$

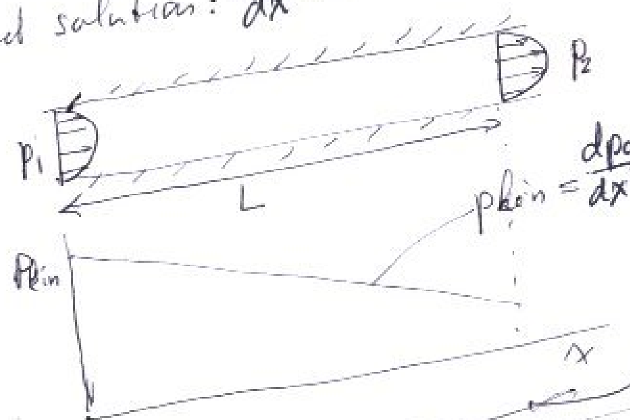
B.C. at $y=0$, $u=0$ $\rightarrow E = 0$

B.C. at $y=h$, $u=0$ $\rightarrow 0 = \frac{dp_0}{dx} \frac{1}{2} h^2 + Dh \rightarrow D = -\frac{1}{2} \frac{dp_0}{dx} h$

Then $\boxed{u = -\frac{1}{2\mu} \frac{dp_0}{dx} (hy - y^2)}$

$\frac{dp_0}{dx} < 0$ needed for flow to the right:

6) Interpret solution:



$p_{kin} = \frac{dp_0}{dx} x + p_1$

$p_{kin_1} - p_{kin_2} = -\frac{dp_0}{dx} L$

$u = \frac{p_{kin_1} - p_{kin_2}}{2 \mu L} (hy - y^2)$, parabolic (6)

$\tau = \tau_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \frac{p_{kin_1} - p_{kin_2}}{2L} (h - 2y)$

$\tau_{y=0} = \frac{p_{kin_1} - p_{kin_2}}{2L} h$ $\quad \tau_{y=h} = -\frac{p_{kin_1} - p_{kin_2}}{2L} h$

mass flow through an area

$\dot{m} = \int \rho \vec{v} \cdot \vec{n} \, dA = \int \rho u \, dy \cdot 1 =$