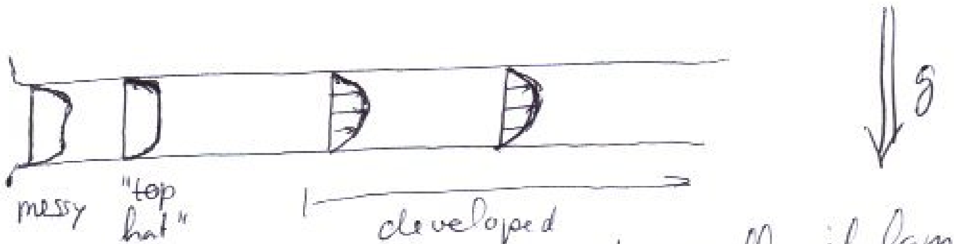
Couette & Plane Poiseuille Flow



The developed region can be solved exactly if laminar!

Assumptions:
1. Incompressible
2. Newtonian
3. Laminar, steady
4. 2D $w = 0$, $\frac{\partial}{\partial z} = 0$
5. velocity is independent of x

Method: (Start with easiest equation)

1) Plug assumptions into continuity, (solve as far as possible:)

$$\cancel{\frac{1}{\rho}\frac{D\rho}{Dt}}^{(1)} + \cancel{\frac{\partial u}{\partial x}}^{(5)} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}}^{(4)} = 0$$

$$\frac{\partial v}{\partial y} = 0 \longrightarrow v = v(\cancel{x}^{(5)}, \cancel{z}^{(4)}, \cancel{t}^{(3)})$$

(Use wall Boundary condition) $v = 0$ at $y = 0$, any x

$$v = v(x) = 0 \text{ any } x \qquad \boxed{v = 0} \quad (6)$$

2) Plug assumptions (and current results) into (y) momentum

$$\rho \cancel{\frac{Dv}{Dt}}^{(6)} = -\frac{\partial p}{\partial y} + \mu \cancel{\nabla^2 v}^{(6)} - \rho g$$

$$\cancel{\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial y}\right)}$$

integration constant!

$$0 = \frac{\partial p}{\partial y} + \rho g \qquad \boxed{\text{Phil} =}\ \boxed{p + \rho g y = p_0(x, \cancel{z}^{(4)}, \cancel{t}^{(3)})} \quad (7)$$

3) Plug assumptions and current results into x-momentum

$$\rho\left(\cancel{\frac{\partial u}{\partial t}}^{(3)} + \cancel{u\frac{\partial u}{\partial x}}^{(5)} + \cancel{v\frac{\partial u}{\partial y}}^{(6)} + \cancel{w\frac{\partial u}{\partial z}}^{(4)}\right) = -\frac{\partial p}{\partial x} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}^{(5)} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}^{(4)}\right)$$

$$0 = -p_0'(x) + \mu \frac{\partial^2 u}{\partial y^2}$$

4) Examine restrictions on the $p_0^{\nabla p}(x)$ pressure ~~constant~~ integration constant:

$$\hat{p}_0^{\nabla p}(x) = \mu \frac{\partial^2 u}{\partial x^2} \text{ is independent of } x \text{ according to } (5)$$

$\rightarrow p_0^{\nabla p}$ is a constant.

$$\rightarrow p_0(x) = \frac{dp_0}{dx} x + p_1$$

$$\rightarrow \boxed{p + \rho g y = \frac{dp_0}{dx} x + p_1} \quad (7a) \quad \boxed{\frac{dp_0}{dx} \text{ constant}} \quad (5) \quad (4) \quad (3)$$

$$u = u(\cancel{x}, y, \cancel{z}, \cancel{t})$$

5) Find u from x-momentum and B.C.

$$\mu \frac{\partial^2 u}{\partial y^2} = \mu \frac{d^2 u}{dy^2} = \frac{dp_0}{dx} \rightarrow \mu \frac{du}{dy} = \frac{dp_0}{dx} y + D \rightarrow \mu u = \frac{dp_0}{dx}\frac{1}{2} y^2 + Dy + E$$

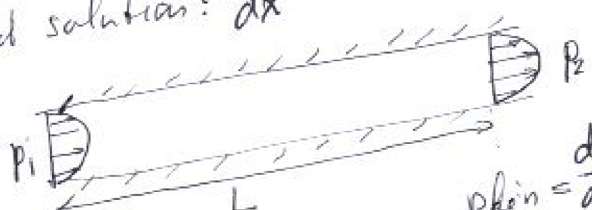
B.C. at $y=0$, $u=0 \rightarrow E=0$

B.C. at $y=h$, $u=0 \rightarrow 0 = \frac{dp_0}{dx}\frac{1}{2}h^2 + Dh \rightarrow D = -\frac{1}{2}\frac{dp_0}{dx}h$

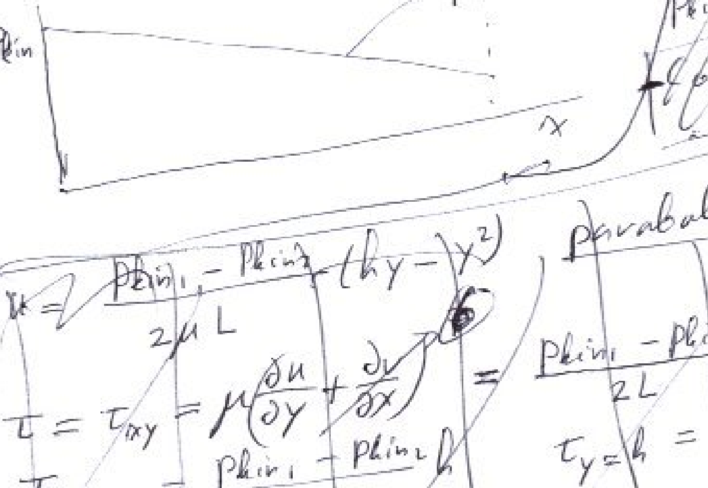
$$\boxed{u = -\frac{1}{2\mu}\frac{dp_0}{dx}(hy - y^2)}$$

Then ~~$u = -\frac{1}{2\mu}(hy - y^2)$~~

6) Interpret solution: $\frac{dp_0}{dx} < 0$ needed for flow to the right:



$$p_{kin} = \frac{dp_0}{dx} x + p_1$$



~~$p_{kin_1} - p_{kin_2} = -\frac{dp}{dx} L$~~

~~$-\frac{dp}{dx} = \frac{p_{kin_1} - p_{kin_2}}{L}$ (synergy effects)~~

$$u = \frac{p_{kin_1} - p_{kin_2}}{2\mu L}(hy - y^2) \quad \text{parabolic}$$

$$\tau = \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{p_{kin_1} - p_{kin_2}}{2L}(h - 2y)$$

$$\tau_{y=0} = \frac{p_{kin_1} - p_{kin_2}}{2L}h \qquad \tau_{y=h} = -\frac{p_{kin_1} - p_{kin_2}}{2L}h$$

mass flow through an area

$$\dot{m} = \int \rho \vec{v} \cdot \vec{n}\, dA = \int \rho u\, dy \cdot 1 =$$

$$\boxed{\frac{dp_{kin}}{dx} = \text{global constant} = \frac{p_{kin_2} - p_{kin_1}}{L}}$$

ignoring entrance effects

$$\boxed{u = -\frac{1}{2\mu}\frac{dp_{kin}}{dx}(hy - y^2)}$$ parabolic, zeros at $y=0$, $y=h$

$$\boxed{u_{max} = -\frac{h^2}{8\mu}\frac{dp_{kin}}{dx}} \rightarrow u = u_{max}\, 4 \frac{y}{h}\left(1 - \frac{y}{h}\right)$$

Mass flow / unit span and time

$$\dot m = \int \rho \underbrace{\vec V \cdot \vec n}_{\vec V \cdot \vec n = u} dA = \int \rho u \, dy \, 1 = \frac{2}{3}\rho u_{max} h$$

$$\boxed{\dot m \equiv \rho u_{ave} h \quad \Big| \quad u_{ave} = \frac{2}{3} u_{max}} \qquad \boxed{u_{ave} = -\frac{h^2}{12\mu}\frac{dp_{kin}}{dx}} \; ~~standard~~$$

$$\boxed{\dot m = -\frac{\rho h^3}{12\mu}\frac{dp_{kin}}{dx}}$$

$$\tau \equiv \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \cancel{\frac{\partial v}{\partial x}}\right) = -\frac{dp_{kin}}{dx}\left(\tfrac{1}{2}h - y\right)$$

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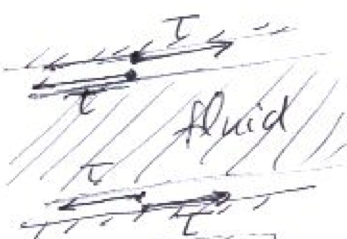
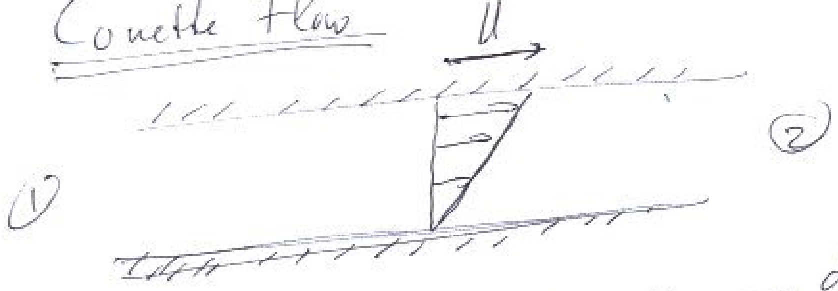


$$\tau = -\frac{dp_{kin}}{dx}\left(\tfrac{1}{2}h - y\right)$$

$$\boxed{\tau_{y=0} = -\frac{1}{2}h\frac{dp_{kin}}{dx} \qquad \tau_{y=h} = \frac{1}{2}h\frac{dp_{kin}}{dx}}$$

The shear force on a ~~unit~~ length L of fluid $\rightarrow 2 * \tau_{y=0} L$
balances the pressure force: $-\frac{dp_{kin}}{dx} h L$.

Couette Flow



assumptions : $p_1 = p_2 \rightarrow \frac{dp_{lin}}{dx} = 0$

Derivation same, but now

$$\mu u = \cancel{\frac{dp_0}{dx}\frac{1}{2}y^2} + Dy + E$$

$$\left.\begin{array}{l} u = 0 \text{ at } y = 0 \rightarrow E = 0 \\ u = U \text{ at } y = h \rightarrow D = \frac{U}{h} \end{array}\right\} \boxed{u = \frac{U}{h} y}$$

$$\boxed{\tau = \frac{\mu U}{h}}$$