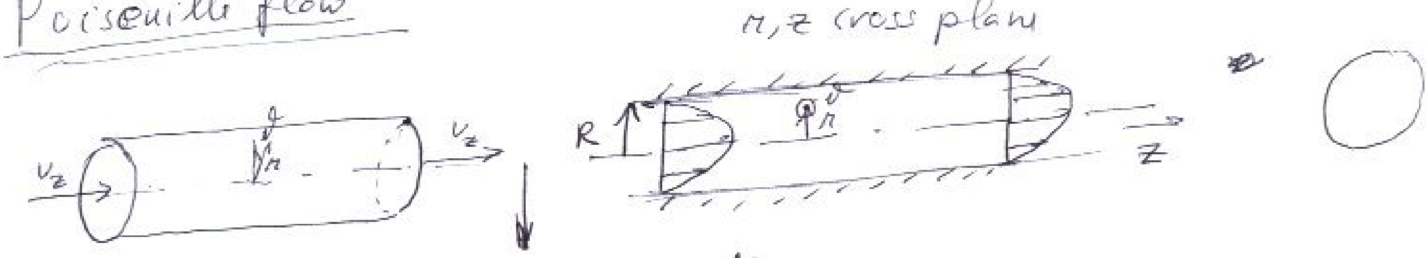
Poiseuille flow



Assumptions

1. Incompressible
2. Newtonian
3. Laminar, steady velocity field
4. velocity is independent of θ
5. velocity is independent of z
6. nonsingular at $r = 0$

Continuity

$$\cancel{\frac{1}{\rho}\frac{D\rho}{Dt}}^{(1)} + \frac{1}{r}\frac{\partial r v_r}{\partial r} + \cancel{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}^{(4)} + \cancel{\frac{\partial v_z}{\partial z}}^{(5)} = 0$$

$$\frac{\partial r v_r}{\partial r} = 0 \rightarrow r v_r = C(\cancel{\theta}^{(4)}, \cancel{z}^{(5)}, \cancel{t}^{(3)})$$

$$v_r = \frac{C}{r}, \text{ B.C. at } r = R \quad \frac{C}{R} = v_r = 0 \rightarrow C = 0$$

$$\rightarrow \boxed{v_r = 0} \text{ everywhere } (7)$$

students
$v_r = 0$ at R.
~~what assumption~~
If so, what assumptions and B.C. are not needed

θ-momentum

$$\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}}^{(3)} + \cancel{v_r}^{(7)}\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\cancel{\frac{\partial v_\theta}{\partial \theta}}^{(4)} + \cancel{\frac{v_r v_\theta}{r}}^{(7)} + v_z \cancel{\frac{\partial v_\theta}{\partial z}}^{(5)}\right)$$

$$= \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r v_\theta}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}^{(4)} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}}^{(5)} + \cancel{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}^{(7),(4)}\right] - \frac{1}{r}\frac{\partial p}{\partial \theta}$$

$$\cancel{\frac{1}{r}\frac{\partial p}{\partial \theta} = \mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r v_\theta}{\partial r}\right) = f(r, \theta, z, t)}$$ ~~doesn't depend on z~~

$$\frac{\partial p}{\partial \theta} = \mu r \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r v_\theta}{\partial r}\right) = f(r, \cancel{\theta}^{(4)}, \cancel{z}^{(5)}, \cancel{t}^{(3)})$$

$$\cancel{= f(r)\theta + C(r, z, t) \rightarrow p(2\pi)}$$

$$p \cancel{= f(r)\theta}$$

$$\cancel{p(2\pi, r, z, t)}$$

$$p(r, \theta, z, t) = f(r)\,\theta + p_0(r, z, t)$$

$$\text{so} \quad p(r, 2\pi, z, t) = p(r, 0, z, t) + f(r)\,2\pi$$

students
~~p = p(~~
what can I say about $p(r, \theta, z, t)$
can ~~it~~ it depend on θ?

But $p(r, 2\pi, z, t) = p(r, 0, z, t)$

so $f(r)$ must be ϕ $\rightarrow p = p_0(r, z, t)$ (8)

also $f(r) = \mu r \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r v_\theta}{\partial r}\right) = 0 \rightarrow \frac{1}{r}\frac{\partial r v_\theta}{\partial r} = C$

$\rightarrow \frac{\partial r v_\theta}{\partial r} = Cr \rightarrow r v_\theta = \frac{1}{2}Cr^2 + D$ D must be ϕ or $v_\theta = \infty$ at $r=0$

$v_\theta = \frac{1}{2}Cr + \cancel{\frac{D}{r}}$

$v_\theta = \frac{1}{2}Cr$, but at $r = R$ $v_\theta|_R = \frac{1}{2}CR = 0 \rightarrow C = 0$

$\rightarrow \boxed{v_\theta \equiv 0}$ (9)

r - momentum

$$\rho\left(\cancel{\frac{\partial v_r}{\partial t}} + \cancel{v_r\frac{\partial v_r}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} - \cancel{\frac{v_\theta^2}{r}} + \cancel{v_z\frac{\partial v_r}{\partial z}}\right) = \mu\left[\cancel{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r v_r}{\partial r}\right)} + \cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_r}{\partial z^2}} - \cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}\right] - \frac{\partial p}{\partial r}$$

(7) (9) (4) (7) (9)

$\rightarrow \frac{\partial p}{\partial r} = 0 \xrightarrow{(8)} p = p_0(z, t)$ (10)

z - momentum

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_r\frac{\partial v_z}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}} + \cancel{v_z\frac{\partial v_z}{\partial z}}\right) = \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right] - \frac{\partial p}{\partial z}$$

(3) (7) (4) (5) (4) (5)

$$\frac{\partial p}{\partial z} = \frac{\partial p_0}{\partial z} = \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right] = f_2(\cancel{z}, \cancel{\theta}, \cancel{z}, \cancel{t}) = \frac{d p_{Rin}}{dz} = P^\nabla$$

(10) (4) (5) (3)

~~$p_0 = f_2(r)z + p_1(t) = p = p(z,t)$~~

$\boxed{p = P^\nabla z + p_1(t) \quad \text{where } P^\nabla \text{ is a global constant}}$

Find v_z $\quad \frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{P^\nabla}{\mu}r \qquad r\frac{\partial v_z}{\partial r} = \frac{P^\nabla}{2\mu}r^2 + A$

becomes infinite at $r=0$ if $A \neq 0$

$\frac{\partial v_z}{\partial r} = \frac{P^\nabla}{2\mu}r + \cancel{\frac{A}{r}}$

$v_z = \frac{P^\nabla}{4\mu}r^2 + B$

$v_z = 0$ at $r = R \qquad \frac{P^\nabla}{4\mu}R^2 + B = 0 \quad B = -\frac{P^\nabla}{4\mu}R^2$

$\boxed{v_z = -\frac{P^\nabla}{4\mu}\left(R^2 - r^2\right)} \qquad v_{z_{max}} = -\frac{P^\nabla}{4\mu}R^2$

$\boxed{\text{parabolic}}$

$v_z = v_{z_{max}}\left(1 - \frac{r^2}{R^2}\right)$

mass flow / unit time

$$\dot{m} = \int \rho\, \vec{U}\cdot\vec{n}\, dA$$

$$= \iiint \rho\, v_z\, r\,dr\,d\theta = 2\pi\rho \int v_z\, r\,dr$$

$$= 2\pi\rho\, R^2\, v_{z\,max} \int_0^1 \left(1 - \frac{r^2}{R^2}\right) \frac{r}{R}\, d\frac{r}{R}$$

$$= 2\pi\rho R^2 v_{z\,max} \left(\frac{1}{2} - \frac{1}{4}\right) = \rho\, \frac{1}{2} v_{z\,max}\, \pi R^2$$

$$\boxed{v_{ave} = -\frac{R^2}{8\mu}\frac{dp}{dz}}$$

$$\boxed{v_{ave} = \frac{1}{2} v_{max}} \qquad \boxed{\dot{m} = \rho\, v_{ave}\, \pi R^2}$$

$\boxed{\frac{dp}{dz}}$ a global constant

$\cancel{v_{ave} = \frac{R^2}{8\mu}}$

Head (pressure) loss

$$-\Delta p = -\frac{dp}{dz} L = L\, \frac{8\mu}{R^2}\, v_{ave}$$

$$= \frac{32\mu}{D^2} L\, v_{ave} = \frac{64\mu}{\rho\, v_{ave} D}\, \frac{L}{D}\, \frac{1}{2} \rho\, v_{ave}^2$$

$$\boxed{h_\ell = -\frac{\Delta p}{\rho} = \frac{64\mu}{\rho\, v_{ave} D}\, \frac{L}{D}\, \frac{1}{2} v_{ave}^2}$$

$\frac{64}{Re_{D,ave}}$ is called the "friction factor" f.

General cylindrical pipe flows

$p_{kin} = p_{kin}(x) = \frac{dp}{dx} x + p_0$ $\qquad \frac{dp}{dx}$ constant

Assume $v = w = 0$ $\qquad u = u(y, z)$

$\cancel{u_x} + \cancel{v_y} + \cancel{w_z} = 0$

$\cancel{\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \mu \nabla^2 v} \qquad \cancel{\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \mu \nabla^2 w}$

$$\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} + \cancel{w\frac{\partial u}{\partial z}} = -\frac{dp}{dx} + \mu \nabla^2 u = -\frac{dp}{dx} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rightarrow \quad \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = +\frac{1}{\mu}\frac{dp}{dx} = \text{~~constant~~, negative}$$

1) $u = -A^2x^2 - B^2y^2 + C$ produces constant second order derivatives and $u = 0$ on an ellipse

$\Rightarrow$ flow in ~~elliptic~~ pipes of elliptic cross sections $\qquad \frac{dp}{dx} = -\frac{1}{\mu}(2A^2 + 2B^2)$

2) $u = -A^2y^2 + Bxy = -A^2(y - Cx)y$ $\qquad \frac{dp}{dx} = -\frac{1}{\mu}2A^2$

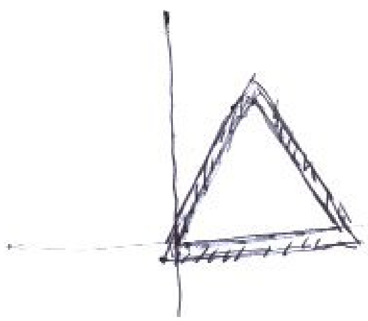
$\Rightarrow$ pipe flow near a corner

3) $u = \overbrace{Cy(y^2 - 3x^2)}^{\text{most general cubic term with } \nabla^2 = 0 \text{, in suitably rotated coordinates}} - A^2 y(y - \sqrt{3}x) + \dots$

$\qquad - A^2 y(y - \sqrt{3}x) + \dots$

$= Cy(y - \sqrt{3}x)(y + \sqrt{3}x) - A^2 y(y - \sqrt{3}x)$

$= y(y - \sqrt{3}x)[-A^2 + C(y + \sqrt{3}x)]$



Flow in ~~an equi~~ a pipe with equilateral triangular cross-section

$$\frac{dp}{dx} = -\frac{1}{\mu}2A^2$$