Bernoulli law Kundu 4.9 Panton 7.2

Euler (inviscid) equations

$$\rho \vec{a} = -\nabla p + \rho \vec{g}$$

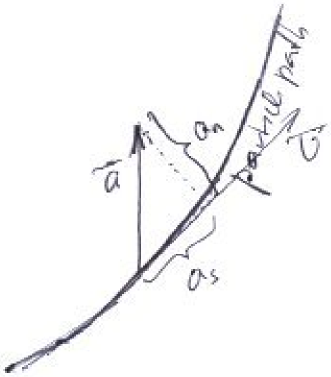
but $\vec{g} = -g \nabla h$ h: height above some reference point (like sea level)

$$\vec{a} = -\frac{1}{\rho}\nabla p - g\nabla h$$

$$\vec{a} = \frac{D\vec{v}}{Dt} \qquad \vec{v} = \frac{D\vec{r}}{Dt}$$

Physics: Define s as the "arclength" (distance) along the particle path

Then $|\vec{v}| = \frac{Ds}{Dt}$



Also, the component of $\vec{a}$ along the particle path (or velocity)

$$\boxed{a_s \equiv \vec{a}\cdot\frac{\vec{v}}{|\vec{v}|} = \frac{D^2 s}{Dt^2}}$$

unit vector in the direction of the path

For the component of $\vec{a}$ normal to the particle path

$$\boxed{a_n = a_{centripetal} = \frac{|\vec{v}|^2}{R}}$$ where R is the ~~normal~~ radius of curvature of the particle path

Note $a_n = \vec{a}\cdot\vec{n}$ where $\vec{n} = \frac{D\vec{v}/|\vec{v}|}{|\vec{v}|Dt}$

The acceleration is towards the "center" of curvature (in the direction of positive $\vec{n}$)

~~Bernoulli law in its simplest form~~

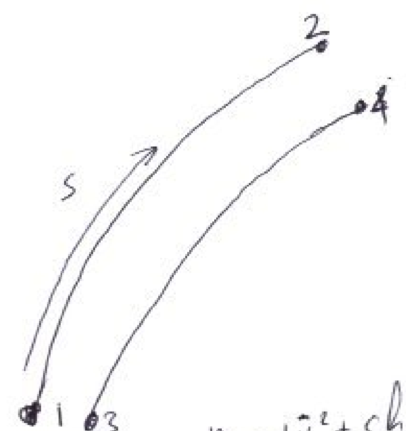
Assumptions: (1) steady
(2) inviscid
(3) along a streamline

Ignoring gravity

$$\boxed{a_n = \frac{\vec{v}^2}{R} = -\frac{\partial p}{\partial n} \quad \text{inviscid??}}$$

R → particle path radius

Bernoulli law in its simplest form



Assumptions:
1. steady
2. inviscid
3. along a single streamline
4. $\rho \equiv$ constant
5. $g \equiv$ constant

Then: $\boxed{\frac{p}{\rho} + \frac{1}{2}\vec{v}^2 + gh = \text{Constant (streamline)}}$ → i.e. $\frac{p_1}{\rho} + \frac{1}{2}\vec{v}_1^2 + gh_1 = \frac{p_2}{\rho} + \frac{1}{2}\vec{v}_2^2 + gh_2 \neq \frac{p_3}{\rho} + \frac{1}{2}\vec{v}_3^2 + gh_3 = \frac{p_4}{\rho} + \frac{1}{2}\vec{v}_4^2 + gh_4$

Not ~~a new~~ an independent equation: follows from momentum

Derivation:

$$a_s = \frac{D|\vec{v}|}{Dt} = \frac{|\vec{v}|\,D|\vec{v}|}{|\vec{v}|\,Dt} = \frac{|\vec{v}|\,D|\vec{v}|}{Ds} = \frac{D\frac{1}{2}\vec{v}^2}{Ds}$$

$(\nabla p)_s = \frac{\partial p}{\partial s}$ = derivative of p along the path direction keeping time constant. Same for $(\nabla h)_s$

Since steady $\frac{D}{Ds} = \frac{\partial}{\partial s} = \frac{d}{ds}$

So $\frac{d\left(\frac{1}{2}v^2 + \frac{p}{\rho} + gh\right)}{ds} = 0$ ~~constant~~ along the streamline

making $\frac{p}{\rho} + \frac{1}{2}v^2 + gh$ a constant along the line

Extensions:

Compressible: Compressible ~~flow~~ "inviscid" flow (no friction or heat conduction): always → $\frac{p}{\rho}$ becomes enthalpy $\bar{h}$, gh becomes gravitational/unit mass potential V_g

$\rho = \rho(p, s)$ But s is constant along the streamline

Derivation → $\rho = \rho^*(p)$ along the streamline

→ $\int \frac{dp}{\rho}$ exists. In particular, from thermo,

$$T\,\cancel{ds}^{\,0} = du + p\,dv = d\bar{h} - v\,dp = d\bar{h} - \frac{1}{\rho}dp = 0$$

$$\rightarrow \frac{dp}{\rho} = d\bar{h} \quad \rightarrow \frac{1}{\rho}\frac{dp}{ds} = \frac{\partial \bar{h}}{\partial s}$$

Also if $g = g(h)$ (unlikely)

$\int g\,dh$ is still the gravitational potential change dV_g so $g\frac{dh}{ds} = \frac{dV_g}{ds}$

Bernoulli becomes

$$\boxed{\bar{h} + \tfrac{1}{2}\vec{v}^2 + V_g = \text{constant (streamline)}}$$

$\bar{h}$ enthalpy

Steady, inviscid, non conducting

Really an energy equation now

i.e. $\frac{p}{\rho} \to \bar{h}$

$gh \to V_g$

Not just along a streamline

Lamb-Gromeko: vorticity $= \nabla \times \vec{v}$

$$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + \nabla \tfrac{1}{2}\vec{v}^2 - \vec{v}\times\vec{\omega}$$

Assuming steady

$$\nabla \tfrac{1}{2}\vec{v}^2 - \vec{v}\times\vec{\omega} = -\nabla \overbrace{\frac{p}{\rho}}^{\text{or } \bar{h}} - \nabla \overbrace{gh}^{\text{or } V_g}$$

$$\nabla\underbrace{\left(\frac{p}{\rho} + \tfrac{1}{2}\vec{v}^2 + gh\right)}_{\text{Bernoulli function } B} = \vec{v}\times\vec{\omega}$$

V_g

$\nabla B = \vec{v}\times\vec{\omega}$ normal to both $\vec{v}$ and $\vec{\omega}$

$\rightarrow$ $\vec{v}$ and $\vec{\omega}$ are in the plane of constant B

$\rightarrow$ B also constant along ~~lines~~ vorticity lines.

Not just steady, not just along a streamline

For some flows $\vec{v} = \nabla\varphi$ where φ is called "velocity potential"

Then $\vec{\omega} = \nabla\times\vec{v} = \nabla\times\nabla\varphi = 0$ and Lamb-Gromeko produces

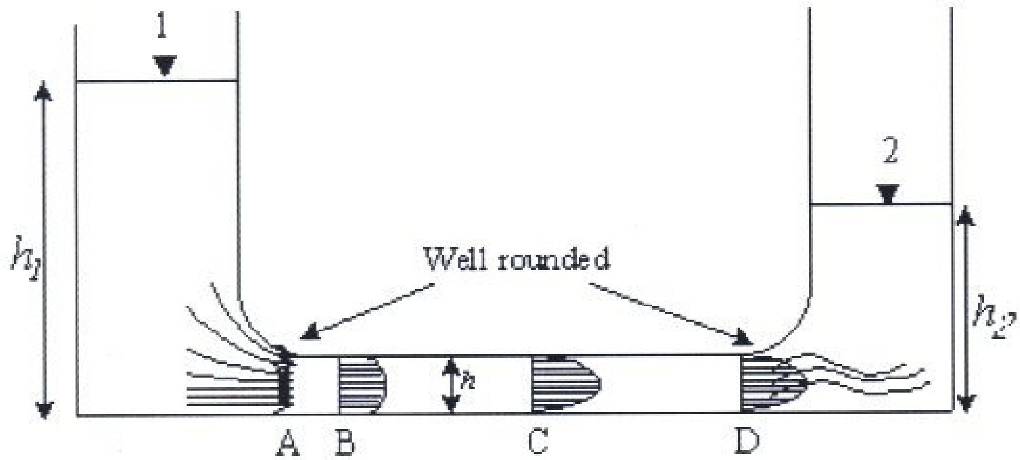
$$\frac{\partial\nabla\varphi}{\partial t} + \nabla\tfrac{1}{2}\vec{v}^2 = -\nabla\overbrace{\frac{p}{\rho}}^{\text{or } \bar{h}} - \nabla\overbrace{gh}^{\text{or } V_g}$$

$$\boxed{\varphi_t + \tfrac{1}{2}\vec{v}^2 + \overbrace{\frac{p}{\rho}}^{\text{or } \bar{h}} + \overbrace{gh}^{\text{or } V_g} = C(t)}$$

Potential flow, inviscid? non-conducting?

Head Loss

1 Duct Flow



Assumptions: $h_1 > h_2$. The reservoirs are wide enough that the flow is quasi-steady. The two reservoirs have the same width, so that $V_1 = V_2$.
Bernoulli:

$$p_1 + \frac{1}{2}\rho V_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho V_2^2 + \rho g h_2$$

Since $p_1 = p_2 = p_a$ and $V_1 = V_2$:

$$\rho g h_1 = \rho g h_2$$

hence $h_1 = h_2$.
Exercise:

What is wrong in this analysis?

1. Is $p_1 = p_2 = p_a$ correct?
2. Is $V_1 = V_2$ correct?
3. When does the Bernoulli law apply?
4. How about the energy balance?
5. Is the Bernoulli law the correct one?
6. Is the length of the connecting duct AD important?

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