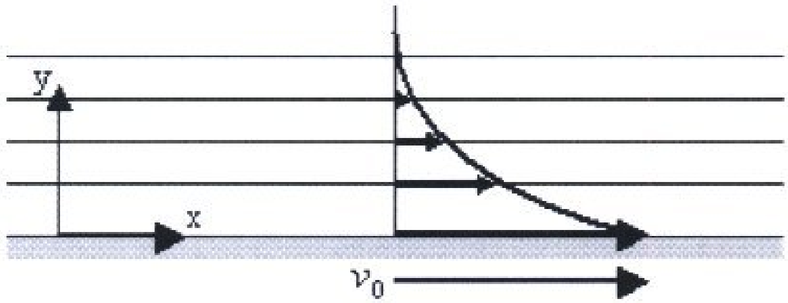
Stokes First Problem

1 Stokes' 1st

Stokes' first problem, also erroneously called Rayleigh flow:



Continuity:

$$\operatorname{div}(\vec{v}) = 0 = \frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}} \implies u = u(y,t)$$

y-momentum:

$$\rho \cancel{\frac{Dv}{Dt}} = -\rho g - \frac{\partial p}{\partial y} + \mu \nabla^2 \cancel{v} \implies p = -\rho g y + P(\cancel{x}, t)$$

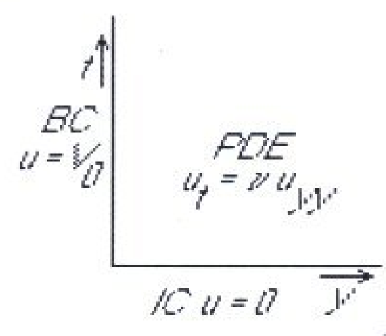
x-momentum:

$$\rho \frac{\partial u}{\partial t} + \rho u \cancel{\frac{\partial u}{\partial x}} + \cancel{\rho v} \frac{\partial u}{\partial y} = -\cancel{\frac{\partial p}{\partial x}} + \mu \left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} \right)$$

The x-momentum equation becomes:

$$u_t = \nu u_{yy}$$

where $\nu = \mu/\rho$ is the dynamic viscosity.



Simpler way: use dimensional analysis

$u = f(y, t, U, \nu)$

But equation is linear

So $u \propto U$

$\frac{u}{U} = f(y, t, \nu)$

choose $t \Rightarrow T$ and $\nu \to \frac{L^2}{T}$

$y\, t^a \nu^b \to L\, T^a L^{2b} T^{-b}$

$\to b = -\frac{1}{2} \quad a = b = -\frac{1}{2}$

Exercise:

How would you normally find u?

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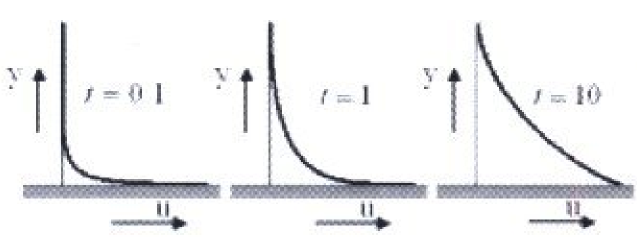
A simpler way to solve ~~is to guess that~~ the solution is *similar*: after rescaling u and y, all velocity profiles look the same.

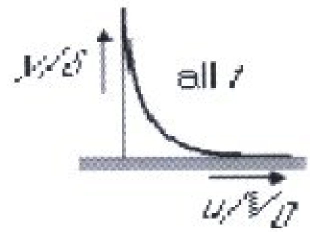
Original profiles:



$\frac{u}{U} = f\left(\frac{y}{\sqrt{4\nu t}}\right)$

$\Pi_y = \eta \equiv \frac{y}{\sqrt{4\nu t}}$

Supposed shape after scaling u with V_0, and y with a characteristic *boundary layer thickness* δ that increases with time:



$$\frac{\partial \eta}{\partial t} \quad \frac{d\eta}{dt}$$

Mathematical form of the similarity assumption:

$$\frac{u}{V_0} = f\left(\frac{y}{\delta(t)}, \cancel{t}\right)$$

$$f_y = \frac{df}{d\eta}\frac{\partial \eta}{\partial y} = \frac{df}{d\eta}\frac{1}{\sqrt{4\nu t}}$$

The proof is in the pudding; if it satisfies the P.D.E., I.C., and B.C., it is OK.

$$u_t = \nu u_{yy} \quad \Longrightarrow \quad -V_0 f' \frac{y}{\delta^2}\delta_t = \nu V_0 f'' \frac{1}{\delta^2}$$

$$\left(\frac{u}{U}\right)_t = \nu \left(\frac{u}{U}\right)_{yy} \qquad \frac{\partial y/\sqrt{4\nu t}}{\partial t} = -\frac{1}{2}\frac{\eta}{t}$$

Put $\eta = y/\delta$:

$$-V_0 f' \eta \frac{\delta_t}{\delta} = \nu V_0 f'' \frac{1}{\delta^2}$$

$$\frac{df}{d\eta}\frac{\partial \eta}{\partial t} = \nu \frac{1}{4\nu t}\frac{d^2 f}{d\eta^2}$$

Separate into terms depending only on η and terms depending only on t:

$$-\frac{1}{2}\frac{\eta}{t} f' = \frac{1}{4t} f''$$

$$-\frac{f'\eta}{f''} = \frac{\nu}{\delta\delta_t} = \text{constant} = \frac{1}{2}$$

It does not make a difference what you take the constant; this merely changes the value of δ, ~~not the~~ physical solution.

Solving the O.D.E.s for δ and f, we solve the P.D.E. For the boundary layer thickness $\delta\delta_t = 2\nu$ so

$$\delta = \sqrt{4\nu t}$$

For the velocity profile $f'' = -2\eta f'$ hence

$$f = \text{erfc}(\eta)$$

where erfc is the *complementary error function* defined as

$$\boxed{\text{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-\xi^2}\,d\xi}$$

Exercise:

Derive the expressions for δ and f.

- Total:

$$\boxed{u = V_0 \,\text{erfc}\left(\frac{y}{\delta}\right) \qquad \delta = \sqrt{4\nu t}}$$

You should now be able to do 7.14, 16, 17