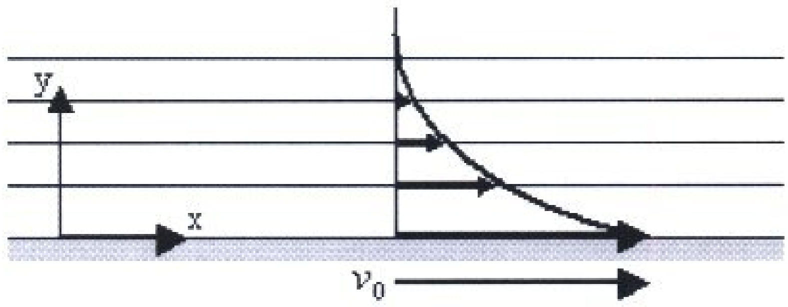
Stokes First Problem

1 Stokes' 1st

Stokes' first problem, also erroneously called Rayleigh flow:



Continuity:

$$\operatorname{div}(\vec{v}) = 0 = \frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}} \implies u = u(y, t)$$

y-momentum:

$$\rho \cancel{\frac{Dv}{Dt}} = -\rho g - \frac{\partial p}{\partial y} + \mu \nabla^2 \cancel{v} \implies p = -\rho g y + P(\cancel{x}, t)$$

x-momentum:

$$\rho \frac{\partial u}{\partial t} + \rho u \cancel{\frac{\partial u}{\partial x}} + \cancel{\rho v} \frac{\partial u}{\partial y} = -\cancel{\frac{\partial p}{\partial x}} + \mu \left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} \right)$$

The x-momentum equation becomes:

$$u_t = \nu u_{yy}$$

where $\nu = \mu/\rho$ is the dynamic viscosity.

Simpler way: use dimensional analysis

$u = f(y, t, U, \nu)$

But equation is <u>linear</u>

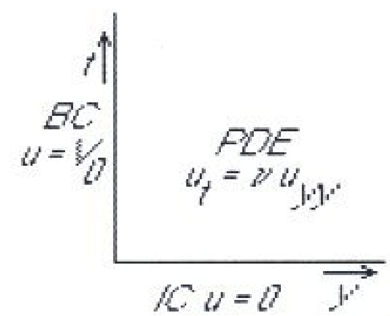
So $u \propto U$

$\frac{u}{U} = f(y, t, \nu)$

choose $t \Rightarrow T$ and $\nu \to \frac{L^2}{T}$

$y t^a \nu^b \to L T^a L^{2b} T^{-b}$

$\to b = -\frac{1}{2}$ $a = b = -\frac{1}{2}$



Exercise:

How would you normally find u?

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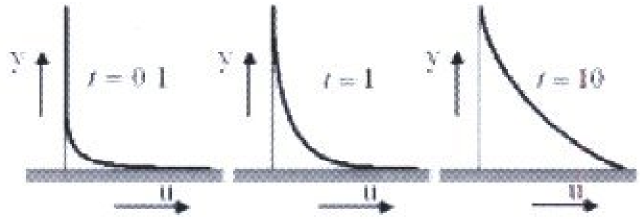
A simpler way to solve ~~is to guess that the~~ the solution is *similar*: after rescaling u and y, all velocity profiles look the same.

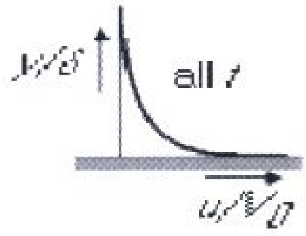
Original profiles:



$\to \frac{u}{U} = f\left(\frac{y}{\sqrt{4\nu t}}\right)$

$\Pi_y = \eta \equiv \frac{y}{\sqrt{4\nu t}}$

Supposed shape after scaling u with V_0, and y with a characteristic *boundary layer thickness* δ that increases with time:



$$\frac{\partial \eta}{\partial t} \quad \frac{\partial \eta}{\partial t}$$

Mathematical form of the similarity assumption:

$$\frac{u}{V_0} = f\left(\frac{y}{\delta(t)}, \cancel{t}\right)$$

$$f_y = \frac{df}{d\eta}\frac{\partial \eta}{\partial y} = \frac{df}{d\eta}\frac{1}{\sqrt{4\nu t}}$$

The proof is in the pudding; if it satisfies the P.D.E., I.C., and B.C., it is OK.

$$u_t = \nu u_{yy} \quad \Longrightarrow \quad -V_0 f' \frac{y}{\delta^2}\delta_t = \nu V_0 f'' \frac{1}{\delta^2}$$

$$\left(\frac{u}{U}\right)_t = \nu \left(\frac{u}{U}\right)_{yy} \qquad \frac{\partial\, y/\sqrt{4\nu t}}{\partial t} = -\frac{1}{2}\frac{\eta}{t}$$

Put $\eta = y/\delta$:

$$-V_0 f' \eta \frac{\delta_t}{\delta} = \nu V_0 f'' \frac{1}{\delta^2}$$

$$\frac{df}{d\eta}\frac{\partial \eta}{\partial t} = \nu \frac{1}{4\nu t}\frac{d^2 f}{d\eta^2}$$

Separate into terms depending only on η and terms depending only on t:

$$-\frac{f'\eta}{f''} = \frac{\nu}{\delta\delta_t} = \text{constant} = \frac{1}{2}$$

$$-\frac{1}{2}\frac{\eta}{t} f' = \frac{1}{4t} f''$$

It does not make a difference what you take the constant; this merely changes the value of δ, ~~not the~~ physical solution.
Solving the O.D.E.s for δ and f, we solve the P.D.E. For the boundary layer thickness $\delta\delta_t = 2\nu$ so

$$\delta = \sqrt{4\nu t}$$

For the velocity profile $f'' = -2\eta f'$ hence

$$f = \mathrm{erfc}(\eta)$$

erf(1) = .8427

where erfc is the *complementary error function* defined as

$$\boxed{\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-\xi^2}\,\mathrm{d}\xi}$$

erfc(1) = .16

Exercise:

Derive the expressions for δ and f.

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Total:

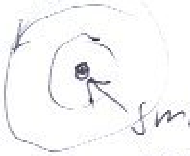
$$\boxed{u = V_0\,\mathrm{erfc}\left(\frac{y}{\delta}\right) \qquad \delta = \sqrt{4\nu t}}$$

You should now be able to do 7.14, 16, 17

Oseen vortex

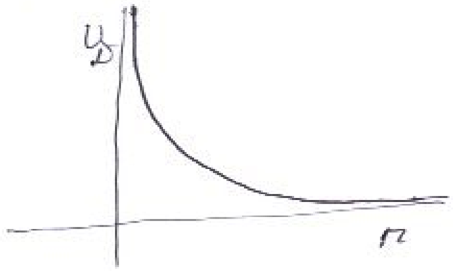
$\boxed{t<0}$

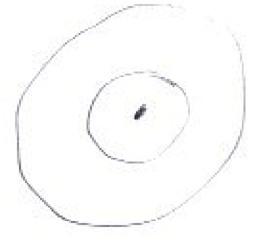
$\Omega \gg 1$



velocity field: $\vec{v} = \frac{\Gamma}{2\pi r}\hat{i}_\vartheta$

(where $\frac{\Gamma}{2\pi\varepsilon} = \Omega\varepsilon$ so $\Omega = O\left(\frac{1}{\varepsilon^2}\right)$)





$\boxed{t=0}$: stop cylinder

$\boxed{t>0}$

Appendix B $\quad v_r = v_z = 0 \quad v_\vartheta = v_\vartheta(r, t) \quad p = p(r,t)$

continuity: $0 = 0$

r-momentum: $-\frac{v_\vartheta^2}{r} = -\frac{1}{\rho}\frac{dp}{dr}$

z-momentum $\quad 0 = 0$

ϑ-momentum $\quad \frac{\partial v_\vartheta}{\partial t} = \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\vartheta}{\partial r}\right) - \frac{v_\vartheta}{r^2}\right]$

$\frac{r}{\nu}\frac{\partial v_\vartheta}{\partial t} = \frac{\partial}{\partial r}\left(r\frac{\partial v_\vartheta}{\partial r}\right) - \frac{v_\vartheta}{r}$

Linearity $\quad v_\vartheta = \Gamma f(r,t,\nu) \qquad \frac{v_\vartheta}{\Gamma} = f(r,t,\nu)$

$\frac{v_\vartheta}{\Gamma} t^a \nu^b \rightarrow L T^{-1} L^{-2} T \, T^a L^{2b} T^{-b} \qquad b = \frac{1}{2} \quad a = \frac{1}{2}$

$v_\vartheta = \frac{\Gamma}{\sqrt{\nu t}} f\left(\frac{r}{\sqrt{\nu t}}\right) \qquad f(\alpha) \sim \frac{1}{\alpha}$ large α

$\Pi_{v/\Gamma} = \frac{v_\vartheta}{\Gamma}\sqrt{\nu t} = f\left(\frac{r}{\sqrt{\nu t}}\right)$

$\frac{r}{\nu}\frac{\Gamma}{\sqrt{\nu t}}\left[-\frac{1}{2t}f + f'\left(-\frac{1}{2}\frac{\eta}{t}\right)\right] = \frac{\nu}{r}\frac{\Gamma}{\sqrt{\nu t}}\left[r f''\frac{1}{\nu t} + f'\frac{1}{\sqrt{\nu t}} - \frac{f}{r}\right]$

$\frac{\Gamma}{\sqrt{\nu t}}\left[-\frac{1}{2t}f + \ldots\right.$

$* \, r^2$

$$\frac{r}{\nu}\frac{\cancel{\Gamma}}{\sqrt{\nu t}}\left[-\frac{1}{2t}f + f'\left(-\frac{1}{2}\frac{\eta}{t}\right)\right] = \frac{\cancel{\Gamma}}{\sqrt{\nu t}}\left[r f''\frac{1}{\nu t} + f'\frac{1}{\sqrt{\nu t}} - \frac{f}{r}\right]$$

$\times\ r\sqrt{\nu t}$:

$$-\frac{1}{2}\eta^2 f - \frac{1}{2}\eta^3 f' = \eta^2 f'' + \eta f' - f$$

$$\eta^2 f'' + (\eta + \tfrac{1}{2}\eta^3)f' - (1 - \tfrac{1}{2}\eta^2)f = 0$$

$$\underbrace{\eta^2 f'' + \eta f' - f}_{\substack{n(n-1)+n-1 \\ = n^2-1}} + \tfrac{1}{2}\eta^2\underbrace{(\eta f' + f)}_{0 \text{ for } f=\eta^{-1}} = 0$$

$\rightarrow \frac{1}{\eta}$ is a solution.

so try $f = \frac{g}{\eta}$ to simplify

$\eta^2 f'' + \eta f' - f = 0$

for $f = \eta^{\pm 1}$

$$f = \frac{g}{\eta} \rightarrow f' = \frac{g'}{\eta} - \frac{g}{\eta^2} \rightarrow f'' = \frac{g''}{\eta} - \frac{2g'}{\eta^2} + \frac{2g}{\eta^3}$$

$$\eta g'' - 2g' + \cancel{\frac{2g}{\eta}} + g' - \cancel{\frac{g}{\eta}} - \cancel{\frac{g}{\eta}} + \tfrac{1}{2}\eta^2\left(g' - \cancel{\frac{g}{\eta}} + \cancel{\frac{g}{\eta}}\right) = 0$$

$$\eta g'' - g' + \tfrac{1}{2}\eta^2 g' = 0$$

$$\frac{dg'}{g'} = \frac{1 - \frac{1}{2}\eta^2}{\eta}d\eta \qquad \ln g' = \ln\eta - \frac{1}{4}\eta^2 + \ln C$$

$$g' = C\eta e^{-\frac{1}{4}\eta^2}$$

$$g = C' e^{-\frac{1}{4}\eta^2} + D$$

$$f = \frac{C' e^{-\frac{1}{4}\eta^2} + D}{\eta}$$

$$v_\theta = \frac{\Gamma}{\sqrt{\nu t}}\frac{\sqrt{\nu t}}{r}\left[C' e^{-\frac{r^2}{4\nu t}} + D\right] \qquad v_\theta \sim \frac{\Gamma}{2\pi r} \text{ for large } r$$

$$v_\theta = \frac{\Gamma}{r}\left[D + C' e^{-\frac{r^2}{4\nu t}}\right] \qquad v_\theta = 0 \text{ at } r=0 \quad C' = -D$$

$$\rightarrow D = \frac{1}{2\pi}$$

$$v_\theta = \frac{\Gamma}{2\pi r}\left[1 - e^{-\frac{r^2}{4\nu t}}\right]$$