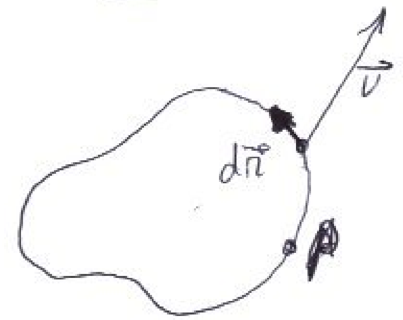
Kelvin theorem

Definition: The "circulation" around a contour (closed curve) in the fluid is defined as:



$$\boxed{\Gamma = \oint_{\substack{C \\ t\ \text{constant}}} \vec{v}\cdot d\vec{r}} \qquad \left(= \int_{\substack{A \\ \text{contour} \\ t\ \text{constant}}}^{A} \vec{v}\cdot d\vec{r}\right)$$

Cylindrical: $d\vec{r} = \hat{i}_r\, dr + \hat{i}_\theta\, r\, d\theta + \hat{i}_z\, dz$

Spherical: $d\vec{r} = \hat{i}_r\, dr + \hat{i}_\theta\, r\, d\theta + \hat{i}_\varphi\, r \sin\theta\, d\varphi$

Cartesian: $d\vec{r} = \hat{i}\, dx + \hat{j}\, dy + \hat{k}\, dz$

Kelvin theorem:

$$\frac{D\Gamma}{Dt} = 0 \quad \text{if}$$

1) A material contour (fluid particles) irrotational incompressible
2) Inviscid (or at least viscous force/unit volume zero)
3) $\rho = \rho(p)$, including $\rho \equiv$ constant or $s \equiv$ constant (inviscid non conducting from uniform source).

Derivation: $\vec{r} = \vec{r}(t;\ \text{particle})$ on a material contour. Take the particle label to be ξ.

$$\Gamma = \oint \vec{v}\cdot d\vec{r} = \oint \vec{v}\, \frac{\partial \vec{r}}{\partial \xi}\, d\xi$$

$$\frac{D\Gamma}{Dt} = \oint \frac{D\vec{v}}{Dt}\frac{\partial \vec{r}}{\partial \xi}\, d\xi + \oint \vec{v}\, \frac{\partial}{\partial \xi}\left(\frac{D\vec{r}}{Dt}\right) d\xi$$

$$= \oint \frac{D\vec{v}}{Dt}\, d\vec{r} + \oint \vec{v}\, \frac{\partial \vec{v}}{\partial \xi}\, d\xi$$

Euler

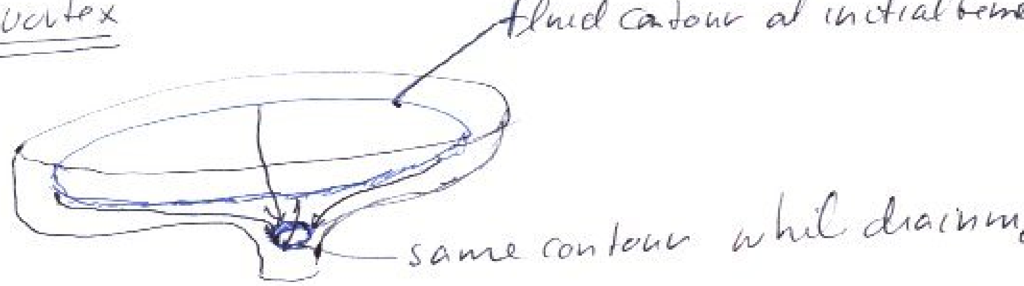
$$= \oint \left[-\frac{1}{\rho}\nabla p + \vec{g}\right] d\vec{r} + \int_A^A \frac{\partial \frac{1}{2}\vec{v}^{\,2}}{\partial \xi}\, d\xi$$

$$= \oint \left[-\nabla \int \frac{dp}{\rho} - g\nabla h\right] d\vec{r} + \frac{1}{2}\vec{v}^{\,2}\Big|_A^A \!\!\to 0$$

$$= \oint -d\left[\int \frac{dp}{\rho}\right] - g\, dh = -\int \frac{dp}{\rho}\Big|_A^A - g h\Big|_A^A = 0$$

Applications

Bath tub vortex

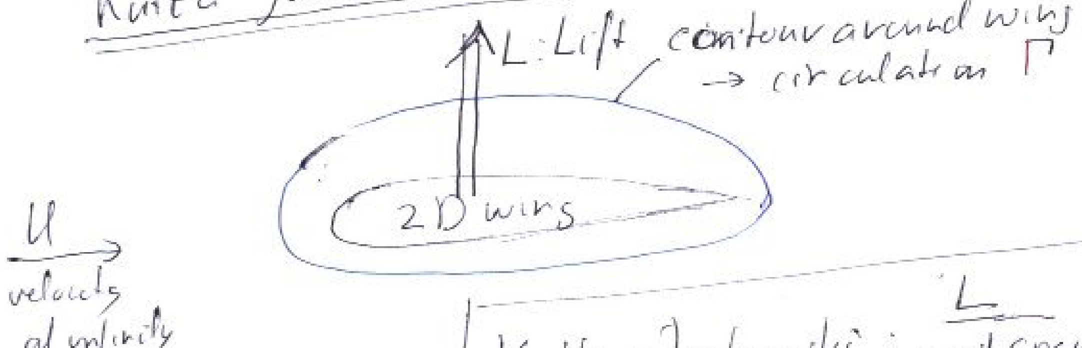


$$\Gamma_{contour} \equiv \oint \vec{v} \cdot d\vec{r} \sim v_{tangential,ave} \times \text{contour length}$$

If contour length becomes much smaller, but Γ remains the same (Kelvin), $v_{tangential,ave}$ must become much bigger

$\rightarrow$ hurricanes, tornados, ... (involve other processes, not barotropic in general)

Kutta Joukowski law



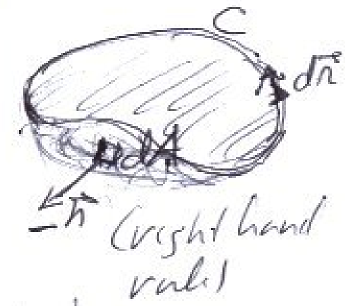
$$\boxed{\text{Kutta–Joukowski: } \frac{L}{\text{unit span}} = \rho U \Gamma}$$

really $-\rho U \Gamma$
(Γ is clockwise for positive lift)

Stokes theorem

$$\Gamma = \oint_C \vec{v} \cdot d\vec{r} = \int_A \vec{\omega} \cdot \vec{n}\, dA$$

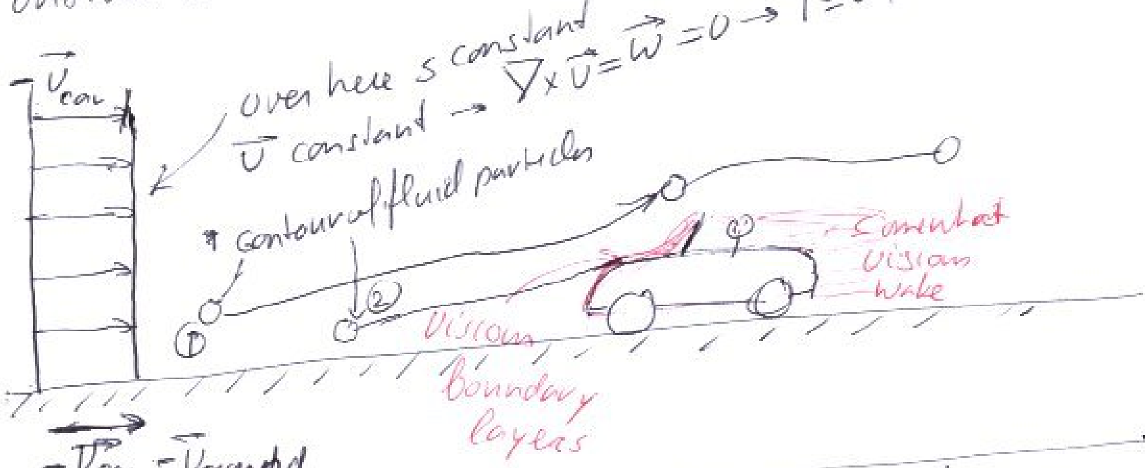
where $\vec{\omega} = \nabla \times \vec{v}$ is the vorticity (in fluids) and A is any area whose perimeter is C and $\vec{n}$ is the unit vector normal to A



So circulation and vorticity are tightly linked!

Applications

Consider a car on a wind-free day



Contour ① stays outside the viscous red region

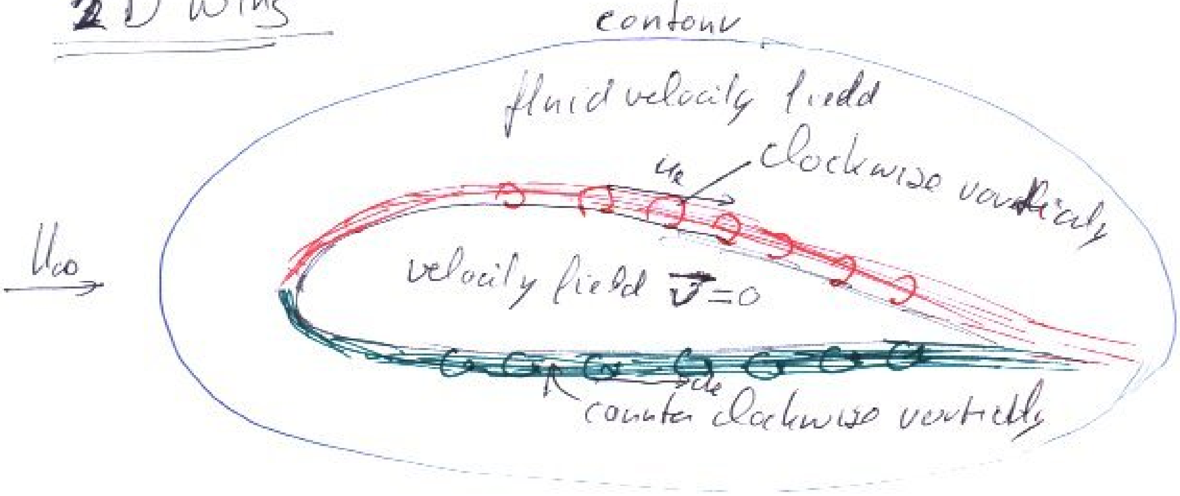
$\to \Gamma$ stays zero $\to \vec{\omega}$ stays zero

Contour ② gets into the boundary layer $\to$ picks up nonzero vorticity

$\Gamma \to$ picks up nonzero vorticity

$\therefore$ Most of a typical car/airplane/... type flow is irrotational ($\nabla \times \vec{v} = \vec{\omega} = 0$)

2D wing



$$\Gamma \underset{\text{Stoke}}{=} \iint \omega_z\,dx\,dy = \iint_{\text{boundary layers}} \omega_z\,dx\,dy$$