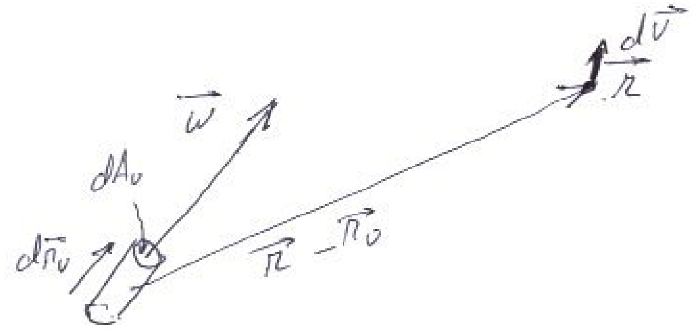
Biot-Savart Law

Finds the velocity given the vorticity.

Consider a little piece $d\vec{r}_v$, dA_v of a vortex tube with circulation $d\Gamma$



$$d\vec{v} = \frac{1}{4\pi} \frac{\vec{\omega}_v \times (\vec{r} - \vec{r}_0)}{|\vec{r} - \vec{r}_v|^3} \, dV_v$$

$\cancel{dV_v = dA_v |d\vec{r}_v|}$

$dV_v = dA_v |d\vec{r}_v|$

$\vec{\omega}_v \, dV_v = d\Gamma \, |d\vec{r}_v|$

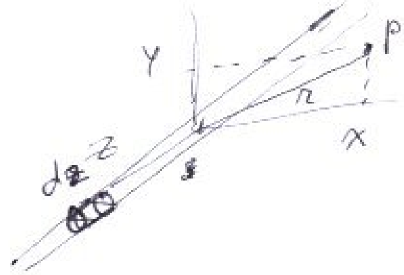
So the streamlines of the little vortex element alone circle around the element

(It is like $\vec{\omega} \times \vec{r}$ in dynamics)

except strength decreases with distance

Applications

2D vortex



$$\vec{v}_p = \frac{1}{4\pi}\int_{z=-\infty}^{+\infty} \frac{\Gamma\, dz\, r\, \hat{i}_\vartheta}{\left(\sqrt{r^2+z^2}\right)^3}$$

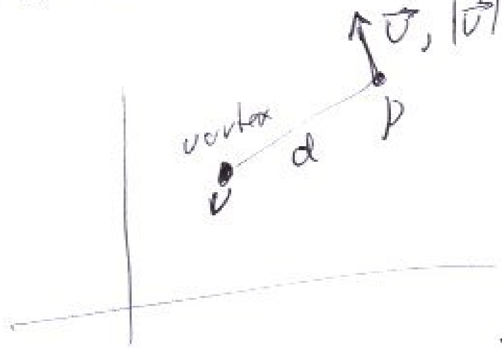
$$= \frac{\hat{i}_\vartheta \Gamma}{4\pi}\int_{-\infty}^{\infty} \frac{r\, dz}{\left(\sqrt{r^2+z^2}\right)^3} \quad \text{put } \frac{z}{r} = u$$

$$= \frac{\hat{i}_\vartheta \Gamma}{4\pi r}\underbrace{\int_{-\infty}^{\infty} \frac{du}{\left(\sqrt{1+u^2}\right)^3}}_{2}$$

$$\boxed{\text{2D vortex along } z\text{-axis: } \quad \vec{v} = \frac{\Gamma}{2\pi r}\hat{i}_\vartheta}$$

Not along z axis but at x_0, y_0: shift

$\vec{v}$, $|\vec{v}| = \frac{\Gamma}{2\pi d}$ $\quad d = |\vec{r}_p - \vec{r}_v|$



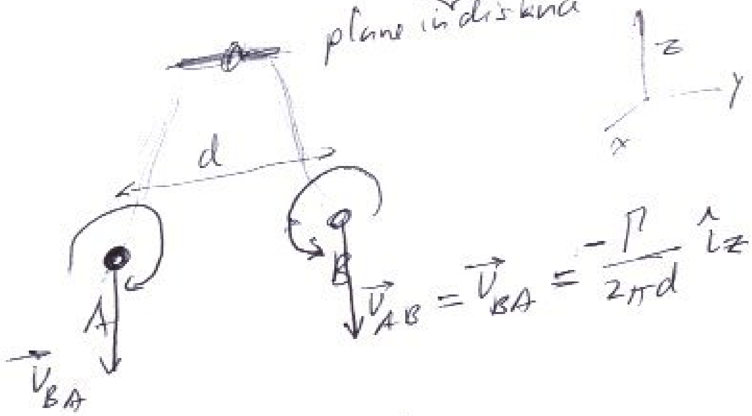
$\cancel{\vec{v} = \frac{\Gamma}{2\pi}\hat{i}_z\, r_p - r_v}$

vector: direction of $\vec{v}$: $\frac{\hat{i}_z \times (\vec{r}_p - \vec{r}_v)}{|\vec{r}_p - \vec{r}_v|}$

$$\rightarrow \vec{v} = \frac{\Gamma}{2\pi}\, \frac{\hat{i}_z \times (\vec{r}_p - \vec{r}_v)}{(\vec{r}_p - \vec{r}_v)^2}$$

Trailing vortices of 3D wings

Look well behind wings → in cross section, the ~~wings a~~ vortices are almost 2D



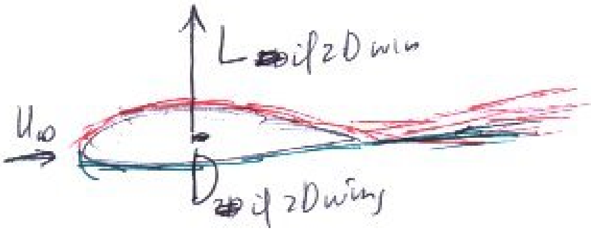
$$\vec{V}_{AB} = \vec{V}_{BA} = \frac{-\Gamma}{2\pi d}\,\hat{i}_z$$

→ Trailing vortices descent

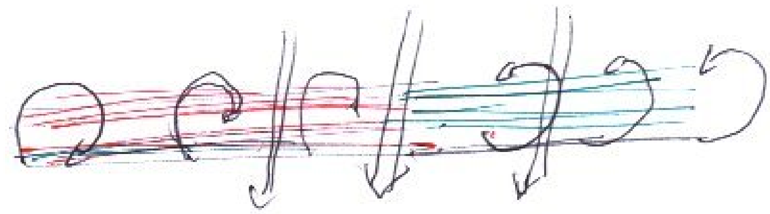
→ Small planes must stay <u>above</u> the flight path of the big Boeing, and touch down <u>after</u> the touch down point of the Boeing -

3D wing

side view:



rear view



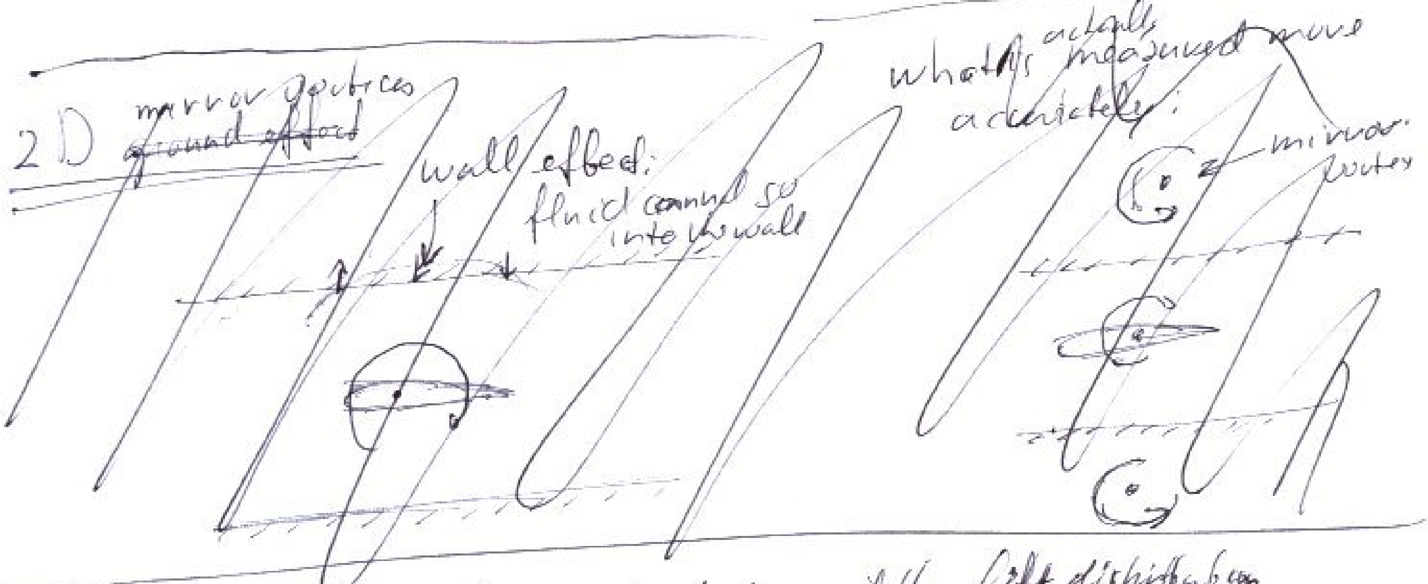
vortices introduce a down draft at the wing

Effective side view

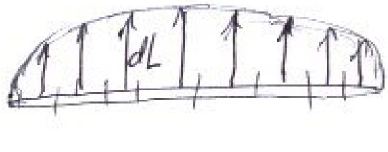
the big 2D L vector rotates to a vector L^*_{2D} turning some of the lift into drag

→ "Induced drag"

2D ~~ground effect~~ mirror vortices

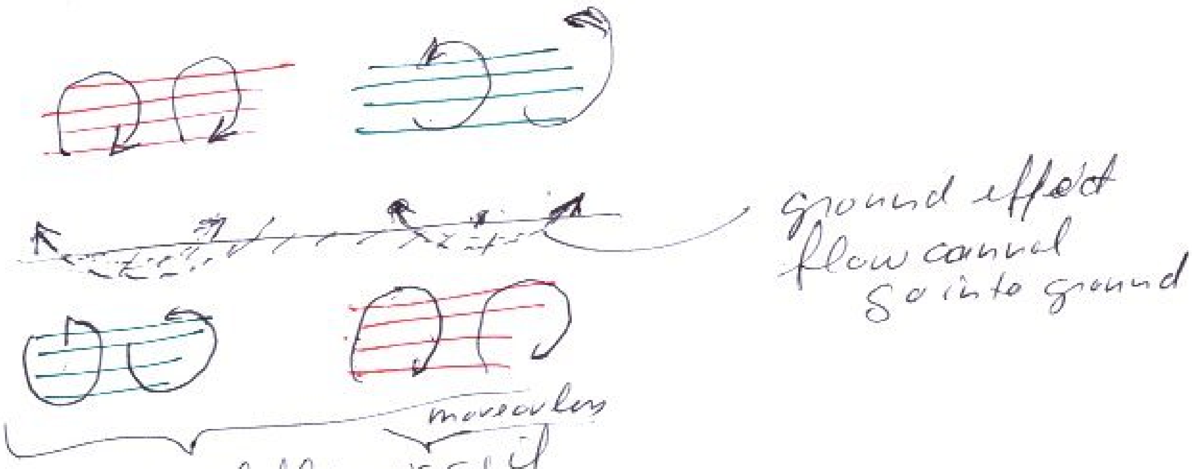


Note: can show: least induced drag if the lift distribution over the span is elliptic



3D ground effect

Rear view of wing



real flow is as if these "mirror" vortices exist in the ground

→ Nearby opposite vortices tend to cancel each others' effects

→ bye induced drag

→ bye landing the plane