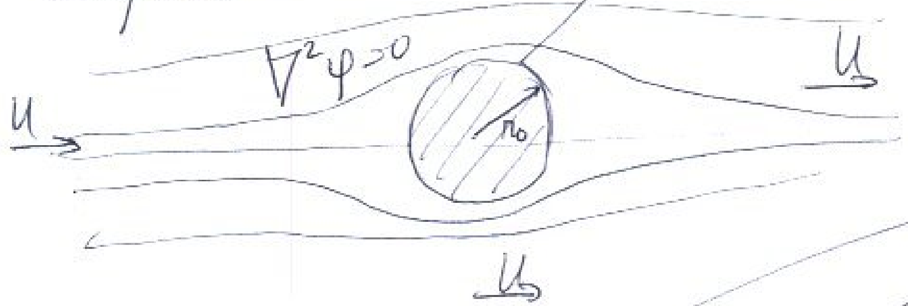
Cylinder flow

$\varphi \approx Ux = Ur\cos\theta$ $\qquad v_r = 0 = \frac{\partial \varphi}{\partial r}$ at $r = r_0$



$$\nabla^2 \varphi = \varphi_{rr} + \frac{1}{r}\varphi_r + \frac{1}{r^2}\varphi_{\theta\theta}$$

Try $\varphi = f(r)\cos\theta$

$$\underbrace{\left[f'' + \frac{1}{r}f' - \frac{1}{r^2}f\right]}\cos\theta = 0$$

$$f'' + \frac{1}{r}f' - \frac{1}{r^2}f = 0 \longrightarrow f = c_1 r + \frac{c_2}{r}$$

So $\varphi = \left(c_1 r + \frac{c_2}{r}\right)\cos\theta$ $\qquad r \to \infty \quad c_1 r\cos\theta = Ur\cos\theta \;\to c_1 = U$

$$\varphi = \left(Ur + \frac{c_2}{r}\right)\cos\theta$$

$$\varphi_r\Big|_{r_0} = \left(U - \frac{c_2}{r_0^2}\right)\cos\theta = 0 \implies c_2 = Ur_0^2$$

$$\boxed{\varphi = U\left(r + \frac{r_0^2}{r}\right)\cos\theta}$$

$\boxed{\text{Streamfunction in 2D incompressible flow}}$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \iff \boxed{u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x}}$$

$$\frac{1}{r}\frac{\partial r v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0 \qquad \boxed{v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad v_\theta = -\frac{\partial \psi}{\partial r}}$$

$\boxed{\text{The lines of constant } \psi \text{ are the streamlines}}$

streamlines $\frac{dy}{dx} = \frac{v}{u} \longrightarrow \frac{dy}{dx} = \frac{-\psi_x}{\psi_y}$ $\qquad \underbrace{\psi_y\, dy + \psi_x\, dx = 0}_{d\psi = 0}$

Boundary conditions are different than for φ: the velocity component normal to the wall is the derivative of ψ in the direction along the wall

$\Rightarrow \boxed{\text{fixed, impenetrable wall} : \ \psi_w = \text{constant}}$

If the flow is also irrotational,

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 = -\frac{\partial \psi_x}{\partial x} - \frac{\partial \psi_y}{\partial y} \ -$$

$\boxed{\nabla^2 \psi = 0 \text{ for incompressible potential flow in 2D}}$