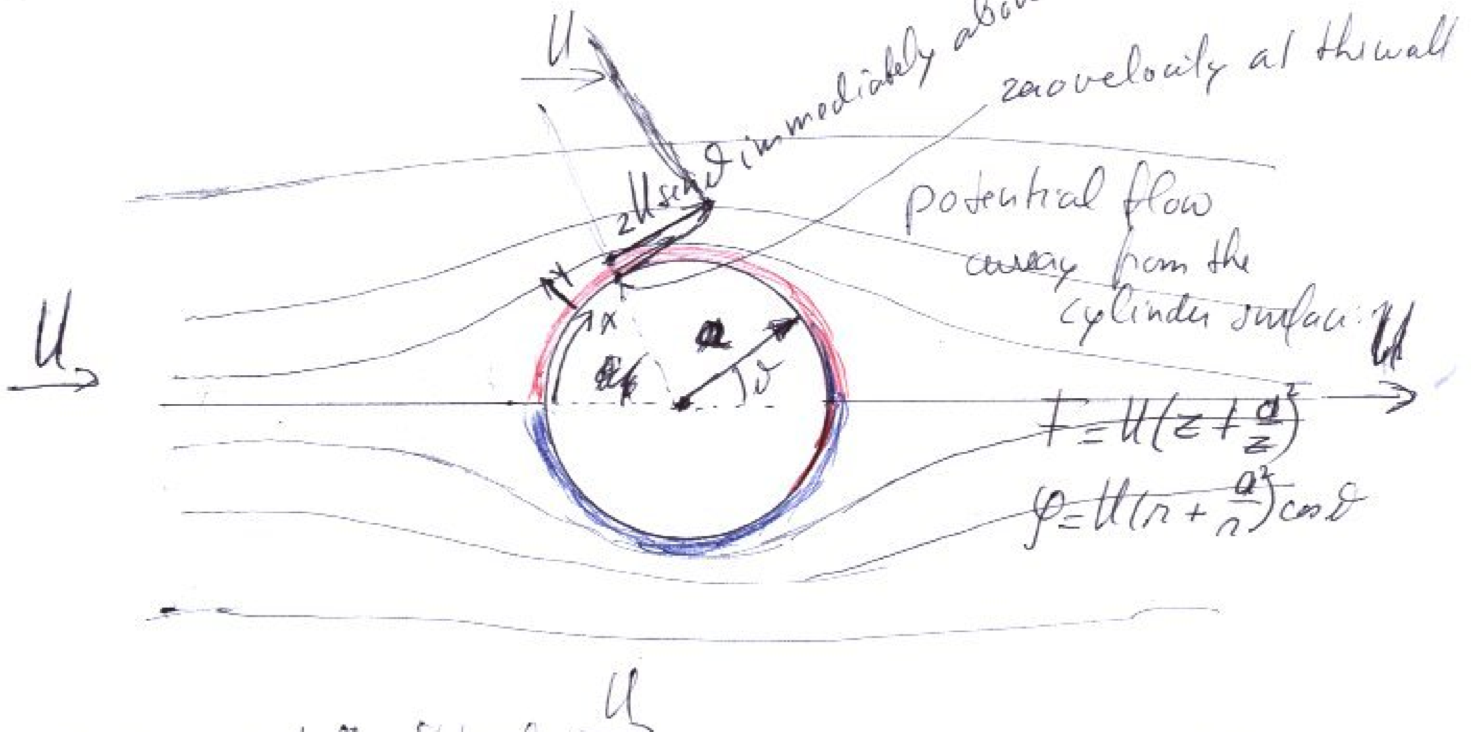
Boundary layers

B2

Example: impulsively started circular cylinder



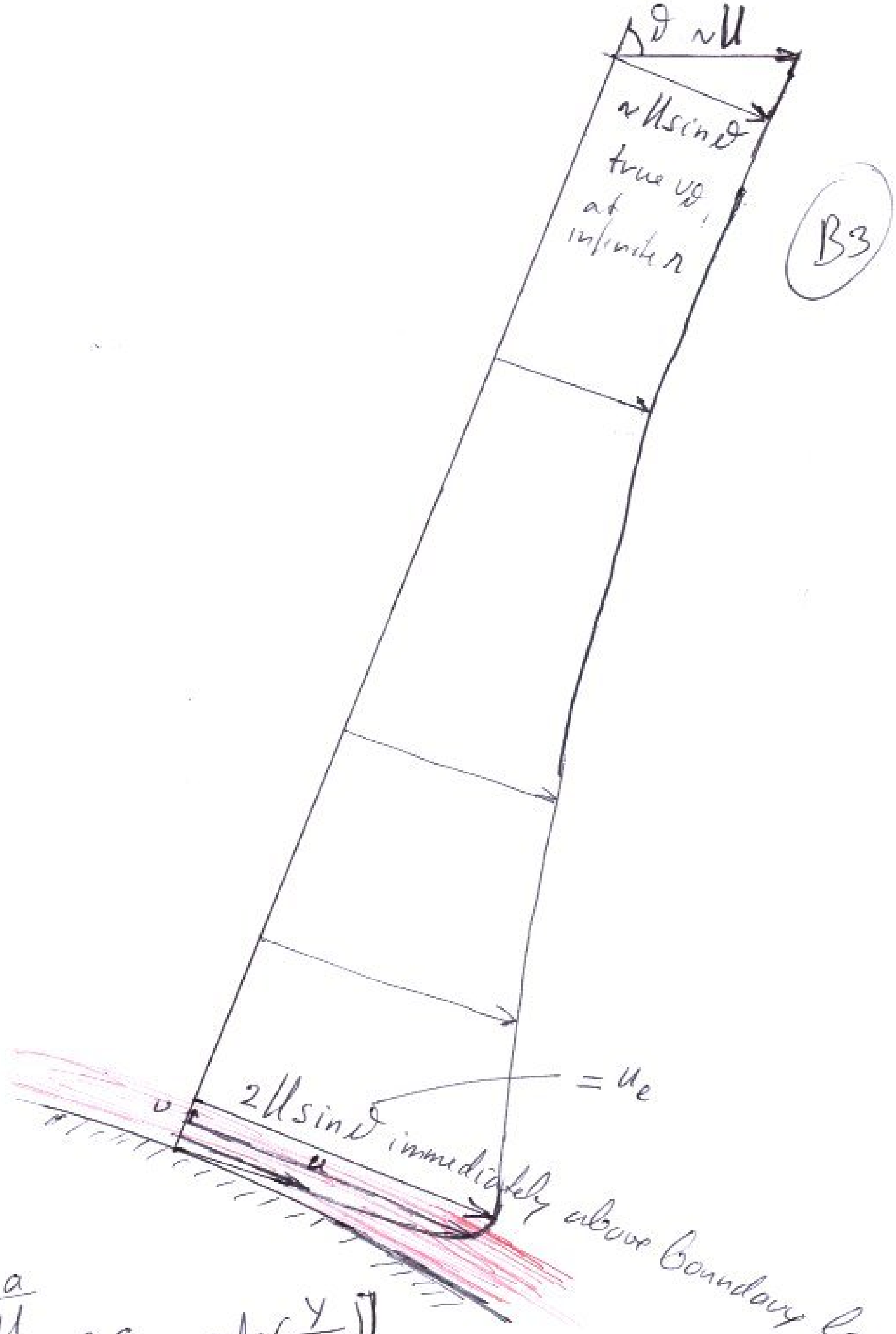
For $t \ll \frac{1}{3}\frac{a}{U}$: Stokes layer

For $t \lesssim \frac{1}{3}\frac{a}{U}$ ($\frac{1}{3}$ radius motion) : no reversed vorticity

For $t \gtrsim \frac{1}{3}\frac{a}{U}$ but $t \lesssim 1.5\frac{a}{U}$: thin layer of reversed vorticity near the rear surface.

For $t \approx 1.5\frac{a}{U}$: Van Dommelen & Shen singularity.

B3

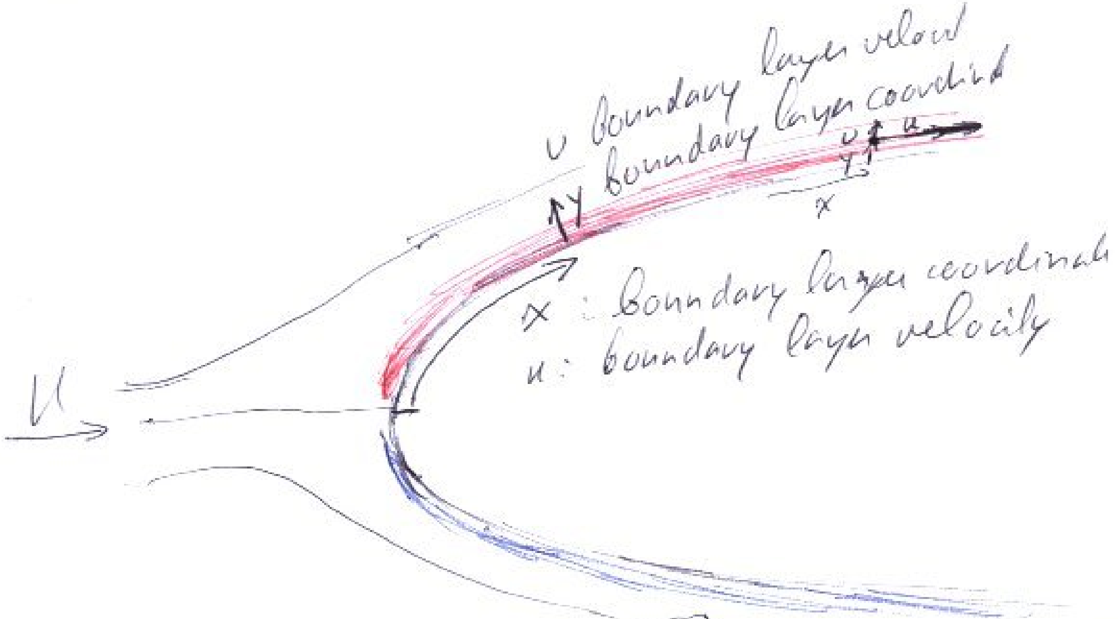


For $t \ll \frac{1}{3}\frac{a}{U}$

$$u \sim 2U\sin\vartheta\left[1 - \mathrm{erfc}\left(\frac{y}{\sqrt{4\nu t}}\right)\right]$$

Boundary Layer Coordinates

Example : parabola (no separation)



Boundary layer coordinates

x : arclength along the wall

y : distance from the wall

u : velocity component parallel to the wall

v : velocity component normal to the wall

NOT Cartesian coordinates (except for a flat plate)

Boundary layer equations

$$Re = \frac{U\ell}{\nu}$$ → typical length

Apply to the limit $Re \to \infty$

B4

Simpler approach: look at limit $\nu \to 0$

Since the x and y coordinates look almost Cartesian on scales comparable to the boundary layer thickness, the equations look Cartesian:

Continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

x - momentum: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

y - momentum: $\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$

But some terms drop out when $\nu \to 0$. Outside the boundary layer, that are the viscous terms. But inside the boundary layer, things change:

here $u = \tilde{u}(x, \frac{y}{\sqrt{\nu}})$ where $\tilde{u}$ is a well behaved function of its argument

$v = \sqrt{\nu}\,\tilde{v}(x, \frac{y}{\sqrt{\nu}})$

$p = \tilde{p}(x, \frac{y}{\sqrt{\nu}})$

where $\tilde{u}, \tilde{v}, \tilde{p}$ are well behaved functions of their arguments (finite derivatives)

so $\frac{\partial u}{\partial y} = \frac{\partial \tilde{u}}{\partial y/\sqrt{\nu}}\frac{\partial y/\sqrt{\nu}}{\partial y} = \frac{\partial \tilde{u}}{\partial y/\sqrt{\nu}}\frac{1}{\sqrt{\nu}}$ $\quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 \tilde{u}}{\partial (y/\sqrt{\nu})^2}\frac{1}{\nu} \to \infty$ when $\nu \to 0$

so $\nu\frac{\partial^2 u}{\partial y^2}$ does not go to zero even if ν does.

(The derivatives of u w.r.t. y are large)

But $\frac{\partial^2 u}{\partial x^2}$ is not large, so $\nu \frac{\partial^2 u}{\partial x^2}$ does still drop out. become negligibly small

$\frac{\partial u}{\partial x}$ finite and $\frac{\partial v}{\partial y} = \sqrt{\nu}\,\frac{\partial \tilde{v}}{\partial y/\sqrt{\nu}}\,\frac{1}{\sqrt{\nu}}$ finite too

Boundary layer equations (2D incompressible)

continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

x-momentum: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho} p_x + \nu \frac{\partial^2 u}{\partial y^2}$

y-momentum: $0 = -\frac{1}{\rho} p_y$

Unfortunately non-linear

~~Can integrate y-momentum immediately~~

~~$p = p(x,t) = p(x,t)$ immediately above the boundary layer~~

~~$= p_t - \frac{1}{2}\rho \vec{v}^2_{\text{wall}}$~~

~~$\approx p_{\text{pot flow, wall}}(x,t)$~~

Can integrate y-momentum equation immediately

$p = p(x,t)$

Let e (external) denote a point just above the boundary layer. Then p inside the boundary layer is

$p = p_e(x,t) \overset{\text{Bernoulli}}{=} p_t - \frac{1}{2}\rho \vec{v}_e^{\,2} \approx p_t - \frac{1}{2}\rho \vec{v}^{\,2}_{\text{pot flow at wall}}$

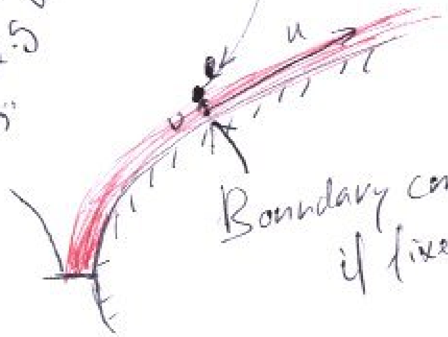
Eg, $p = p_t - \frac{1}{2}\rho(2U\sin\theta)^2$ for impulsively started circular cylinder

Boundary and initial conditions

Initial condition I.C.s: e.g. $u=0$ by symmetry

boundary condition at edge of boundary layer
$u \to u_e$ when $\frac{y}{\sqrt{\nu}} \to \infty$

where u_e is the potential flow velocity for $y \to 0$



Boundary condition at $y=0$:
if fixed solid wall: $u = v = 0$