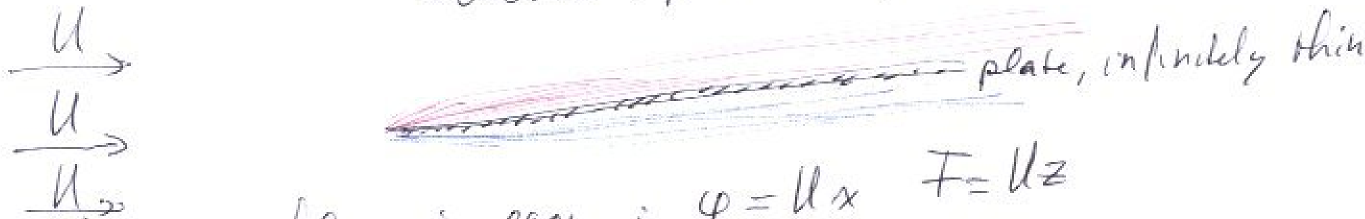
Blasius solution for semi-infinite flat plate

A rare case where the boundary layer coordinates x and y are really Cartesian coordinates

consider top boundary layer only



Potential flow is easy : $\varphi = Ux \quad F = Uz$

$u = U \quad v = 0$

Boundary layer flow:

Use dimensional analysis

$$u = f(x, y, U, \nu)$$

But in boundary layer approximation

$$u = f\left(x, \frac{y}{\sqrt{\nu}}, U\right)$$

Use x and U as selected parameters

$$\Pi_u = \frac{u}{U} \qquad \Pi_{y/\sqrt{\nu}} = \frac{y}{x}\sqrt{\frac{Ux}{\nu}} = y\sqrt{\frac{U}{\nu x}}$$

Apparently $\sqrt{\frac{\nu x}{U}}$ is a typical boundary layer thickness δ

$$\frac{u}{U} = f\left(y\sqrt{\frac{U}{\nu x}}\right)$$

Boundary layer solution for Blasius: B8

Use a streamfunction ψ so that $u = \frac{\partial \psi}{\partial y}$ $\quad v = -\frac{\partial \psi}{\partial x}$

Then continuity reads $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$

automatically.

We take $\psi = 0$ at $y = 0$, then $v = -\frac{\partial \psi}{\partial x} = 0$ at $y = 0$

automatically.

becomes zero in boundary layer approximation, $\nu \to 0$

Now $\quad u = \tilde{u}\left(x, \frac{y}{\sqrt{\nu}}, U, \nu\right)$

$$\psi = \int_{y=0}^{y} \tilde{u}\left(x, \frac{y}{\sqrt{\nu}}, U, 0\right) d\frac{y}{\sqrt{\nu}} \sqrt{\nu}$$

$$\psi = \sqrt{\nu}\, f\left(x, \frac{y}{\sqrt{\nu}}, U\right)$$

Use dimensional analysis. Selected parameters x and U.

$$\Pi_{\psi/\sqrt{\nu}} = \frac{\psi}{Ux}\sqrt{\frac{Ux}{\nu}} = \frac{\psi}{\sqrt{\nu U x}}$$

$$\Pi_{y/\sqrt{\nu}} = \frac{y}{x}\sqrt{\frac{Ux}{\nu}} = \frac{y}{\sqrt{\frac{\nu x}{U}}}$$

Apparently $\delta = \sqrt{\frac{\nu x}{U}}$ is a typical boundary layer thickness, (like $\sqrt{4\nu t}$ in Stokes flow)

$$\Pi_{\psi/\sqrt{\nu}} = f(\Pi_{y/\sqrt{\nu}}) \qquad \frac{\psi}{\sqrt{\nu U x}} = f\left(y\sqrt{\frac{U}{\nu x}}\right)$$

$$\psi = \sqrt{\nu U x}\, f\left(\frac{y}{\sqrt{\nu x/U}}\right)$$

define $\eta = \frac{y}{\sqrt{\nu x/U}}$ $\qquad \eta_y = \frac{1}{\sqrt{\nu x/U}}$ $\qquad \eta_x = -\frac{1}{2}\frac{\eta}{x}$

$$u = \frac{\partial \psi}{\partial y} = f' \frac{\partial \eta}{\partial y} = U f'$$

$$v = -\frac{\partial \psi}{\partial x} = -\sqrt{\nu U x}\left[\frac{1}{2x} f + f' \eta_x\right] = \sqrt{\frac{\nu U}{x}}\left[-\frac{1}{2} f + \frac{1}{2}\eta f'\right]$$

Note $u_e = U$ at edge of boundary layer, since $u = U$ everywhere in the potential flow of Blasius B9

$\rightarrow$ $p_e = p = p_t - \frac{1}{2}\rho U^2$ = constant inside the boundary layer

Plug into
$$\cancel{\frac{\partial u}{\partial t}}^{\text{steady}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\cancel{\frac{\partial p}{\partial x}}^{p\ \text{constant}} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$U f' \, U f''\left(-\frac{1}{2}\frac{\eta}{x}\right) + \sqrt{\frac{\nu U}{x}}\,\frac{1}{2}\left[-f + \eta f'\right] U f'' \sqrt{\frac{U}{\nu x}}$$
$$= \nu U f''' \frac{U}{\nu x}$$

$$f''' = -\frac{1}{2} f f''$$

$$\boxed{f''' + \frac{1}{2} f f'' = 0} \qquad \boxed{u = U f'} \qquad v = \frac{1}{2}\sqrt{\frac{\nu U}{x}}\left[\eta f' - f\right]$$

$\rightarrow$ Numerical solution required

Boundary conditions

$u = 0$ at $\eta = 0$

$v = 0$ at $\eta = 0$

$u = U$ at $\eta = \infty$

$$\boxed{\begin{aligned} f'(0) &= 0 \\ f(0) &= 0 \\ f'(\infty) &= 1 \end{aligned}}$$

Numerical solution: $f''(0) \neq 0$

$$u = U f'\Big(\underbrace{\frac{y}{\sqrt{\nu x / U}}}_{\eta}\Big) \qquad \tau = \mu\frac{\partial u}{\partial y} = \mu\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \mu U f''(\eta)\frac{1}{\sqrt{\nu x / U}}$$

$$\tau_{wall} = \rho\sqrt{\frac{U\nu}{x}}\, U f''(0) \qquad \boxed{C_f = \frac{\tau_{wall}}{\frac{1}{2}\rho U^2} = \frac{2 f''(0)}{\sqrt{Re_x}}} \qquad \boxed{Re_x = \frac{Ux}{\nu}}$$

- Numerical solution: $\boxed{f''(0) = 0.33206}$

Drag force on one side of a plate of length L and unit span:

$$C_D = \frac{F_D}{\frac{1}{2}\rho U^2 L \cdot 1} = \frac{\int_0^L \tau_w \, dx}{\frac{1}{2}\rho U^2 L} = \frac{1}{L}\int_0^L C_f \, dx$$

$$= \frac{1}{L}\int_{x=0}^{L} 2 f''(0) \sqrt{\frac{\nu}{Ux}} \, dx = \frac{2 f''(0)}{L} \sqrt{\frac{\nu}{U}} \, 2\sqrt{x} \Big|_0^L$$

$$\boxed{C_D = \frac{4 f''(0)}{\sqrt{Re_L}} \qquad Re_L = \frac{UL}{\nu}}$$