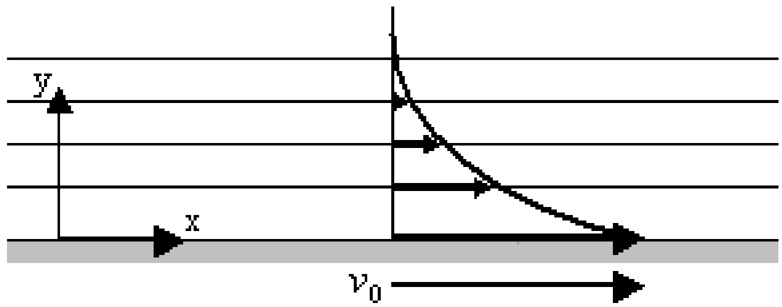
Stokes First Problem

1 Stokes' 1st

Stokes' first problem, also erroneously called Rayleigh flow:



Continuity:

$$\operatorname{div}(\vec{v}) = 0 = \frac{\partial u}{\partial x} + \frac{\partial \cancel{v}}{\cancel{\partial y}} \quad \Longrightarrow \quad u = u(y,t)$$

y-momentum:

$$\rho \frac{D\cancel{v}}{\cancel{D}t} = -\rho g - \frac{\partial p}{\partial y} + \mu \nabla^2 \cancel{v} \quad \Longrightarrow \quad p = -\rho g y + P(\cancel{x}, t)$$

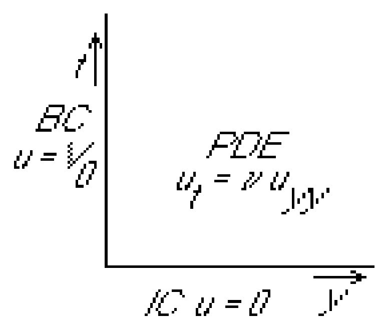
x-momentum:

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial \cancel{u}}{\cancel{\partial x}} + \rho \cancel{v} \frac{\partial u}{\partial y} = -\frac{\partial \cancel{p}}{\cancel{\partial x}} + \mu \left(\frac{\partial^2 \cancel{u}}{\cancel{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} \right)$$

The x-momentum equation becomes:

$$u_t = \nu u_{yy}$$

where $\nu = \mu/\rho$ is the kinematic viscosity.



Exercise:

How would you normally find u?

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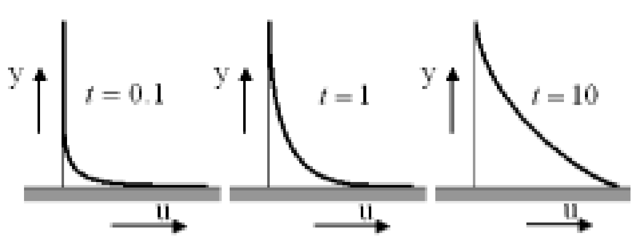
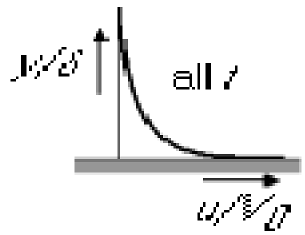
A simpler way to solve is to guess that the solution is *similar*: after rescaling u and y, all velocity profiles look the same.

Original profiles:



Supposed shape after scaling u with V_0, and y with a characteristic *boundary layer thickness* δ that increases with time:



Mathematical form of the similarity assumption:

$$\frac{u}{V_0} = f\left(\frac{y}{\delta(t)}, \cancel{t}\right)$$

The proof is in the pudding; if it satisfies the P.D.E., I.C., and B.C., it is OK.

$$u_t = \nu u_{yy} \quad \Longrightarrow \quad -V_0 f' \frac{y}{\delta^2}\delta_t = \nu V_0 f'' \frac{1}{\delta^2}$$

Put $\eta = y/\delta$:

$$-V_0 f' \eta \frac{\delta_t}{\delta} = \nu V_0 f'' \frac{1}{\delta^2}$$

Separate into terms depending only on η and terms depending only on t:

$$-\frac{f'\eta}{f''} = \frac{\nu}{\delta\delta_t} = \text{constant} = \frac{1}{2}$$

It does not make a difference what you take the constant; this merely changes the value of δ, not the physical solution.
Solving the O.D.E.s for δ and f, we solve the P.D.E. For the boundary layer thickness $\delta\delta_t = 2\nu$ so

$$\delta = \sqrt{4\nu t}$$

For the velocity profile $f'' = -2\eta f'$ hence

$$f = \text{erfc}(\eta)$$

where erfc is the *complementary error function* defined as

$$\boxed{\text{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\xi^2}\, \mathrm{d}\xi}$$

Exercise:

Derive the expressions for δ and f.

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Total:

$$\boxed{u = V_0 \ \text{erfc}\left(\frac{y}{\delta}\right) \qquad \delta = \sqrt{4\nu t}}$$

You should now be able to do 7.14, 16, 17