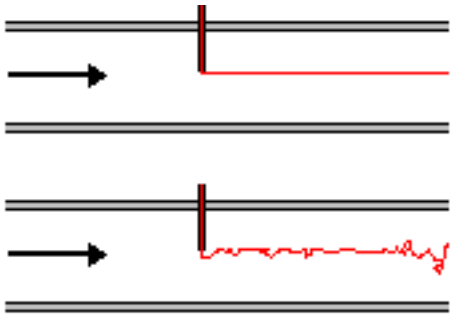
Turbulence

1 Characteristics

(Panton3 26.1,2)

Reynolds experimentally studied laminar pipe flow (Poiseuille flow) using a dye streakline. He showed that above a Reynolds number Re_D (based on the pipe diameter) of roughly 2,300, the flow becomes unsteady and irregular, "turbulent." (By carefully removing disturbances, laminar flow can be maintained at Reynolds numbers that are several times larger.)



For Blasius' flat plate, the "transition" to turbulent flow occurs at a Reynolds number Re_x (based on x) of roughly about 320,000. (For comparison with pipe flow, the Reynolds number based on the boundary layer thickness may be more fair. Taking, rather arbitrarily, $\delta = 5/\sqrt{Re_x}$, Re_δ would be 2,800.) Again laminar flow can be maintained at Reynold numbers several times larger, by very carefully minimizing flow disturbances.

Typical characteristics of turbulent flows are:

- unsteady, irregular 3D vorticity fluctuations;
- large diffusion;
- large Reynolds numbers;
- large dissipation.
- local;
- self-sustaining;

2 Reynolds Decomposition

(Panton3 26.3,4)

The "Reynolds decomposition" is used to analyze turbulent flows that are steady in the average. It decomposes the turbulent flow quantities in average and fluctuating components. First it defines the time averaged flow velocity (indicated with a bar above it) as

$$\boxed{\overline{\vec{v}}\left(\vec{r}\right) = \lim_{T\to\infty}\left(\frac{1}{T}\int_0^T \vec{v}\left(\vec{r},t\right)\,\mathrm{d}t\right)}$$

The fluctuating part of the velocity can now be defined as the remainder:

$$\vec{v}\,' = \vec{v} - \overline{\vec{v}}$$

The total velocity can then be written as

$$\vec{v} = \overline{\vec{v}} + \vec{v}\,'$$

Similarly for the pressure

$$p = \overline{p} + p'$$

Exercise:

> Sketch a velocity or pressure trace in a point in a turbulent pipe flow. In the figure, indicate what the average and fluctuating quantities are. What are the average and fluctuating velocities at the surface of the pipe? Sketch the mean (average) velocity profile in turbulent flow and in laminar flow in the same graph, assuming the two flows have the same net mass flow.

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Reynolds averaged equations can be found by averaging the Navier Stokes equations. In doing so, note that the average of a fluctuation (prime) quantity is always zero, and that averaging an already averaged (bar) quantity leaves the averaged quantity unchanged. Also note that averaging is a linear operation that commutes with addition and differentiation.

Continuity: Averaging the continuity equation gives

$$\overline{\frac{\partial u_i}{\partial x_i}} = 0 \quad \Longrightarrow \quad \overline{\frac{\partial \overline{u_i} + u_i'}{\partial x_i}} = 0$$

As noted above, the average has no effect on the first already averaged term, but the second fluctuating term averages to zero. So you get:

$$\boxed{\frac{\partial \overline{u_i}}{\partial x_i} = 0}$$

This takes the form of the *normal* continuity equation, but applied to the *average* velocity.

Viscous Stress Tensor: Assuming that μ is constant,

$$\overline{\tau_{ij}} = \overline{\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} \quad \Longrightarrow \quad \overline{\tau_{ij}} = \mu\left(\overline{\frac{\partial \overline{u_i} + u_i'}{\partial x_j}} + \overline{\frac{\partial \overline{u_j} + u_j'}{\partial x_i}}\right)$$

or using the same arguments as for continuity

$$\boxed{\overline{\tau_{ij}} = \mu\left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\right)}$$

This takes the form of the *normal* Newtonian stress tensor, but applied to the *average* velocity.

Momentum Equations: The momentum equations (Navier-Stokes equations) are in conservation form and in index notation:

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \partial \tau_{ij} \partial x_j$$

so Reynolds averaging gives

$$\overline{\frac{\partial \rho\left(\overline{u_i} + u_i'\right)}{\partial t}} + \overline{\frac{\partial \rho\left(\overline{u_i} + u_i'\right)\left(\overline{u_j} + u_j'\right)}{\partial x_j}} = -\overline{\frac{\partial \overline{p} + p'}{\partial x_i}} + \overline{\frac{\partial \overline{\tau_{ij}} + \tau_{ij}'}{\partial x_j}}$$

Now note here in the second term that the average of a *product* of fluctuating quantities is not zero. For example, $u_i' u_i'$, a square, is obviously always positive, so its average must be positive too. Taking that into account, you get for the momentum equations:

$$\frac{\partial \rho \overline{u_i}\,\overline{u_j}}{\partial x_j} + \frac{\partial \rho \overline{u_i' u_j'}}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial \overline{\tau_{ij}}}{\partial x_j}$$

The second term above is *not* in the normal momentum equations, It is seen from comparing this additional term with the final term above that it acts as an additional viscous force per unit volume. In that picture,

$$\boxed{\tau_{ij}^{\mathrm{R}} \equiv -\rho \overline{u_i' u_j'}}$$

is the stress, called the "Reynolds stress."

In those terms the momentum equations take the form:

$$\boxed{\frac{\partial \rho \overline{u_i}\,\overline{u_j}}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial \overline{\tau_{ij}} + \tau_{ij}^{\mathrm{R}}}{\partial x_j}}$$

That is like the normal steady Navier Stokes equations, but for the *average* velocity and pressure, and including the additional Reynolds stress.

Unfortunately, the Reynolds stress is not known unless you solve the full unsteady Navier-Stokes equations. To avoid this humongous task, guessing (also known as modelling) is needed.

Exercise:

> Compare the sizes of the Reynolds stress and the laminar stress in high Reynold number flows. In the turbulent pipe flow, is the Reynolds stress everywhere much larger than the laminar stress? If not, at what locations is the laminar stress larger?

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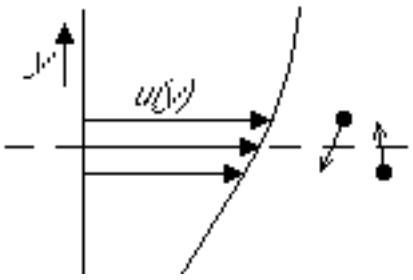
3 Molecular model of (laminar) viscosity

(Panton3 6.2)



To understand the Reynolds stress more clearly, you must first understand how the normal laminar shear stress arises from molecular motion. To do so, look at the steady unidirectional laminar flow sketched in the

figure above. In particular, consider the net momentum transfer due to molecular motion through the plane indicated by the broken line. Let's assume the fluid is a gas. The molecules of the gas are always in random molecular heat motion. On average, they travel a small distance, called the "free-path length" λ, between collisions with other molecules. The velocity with which they travel between collisions is of the order of the speed of sound a.

Due to the random motion, there will always be molecules crossing the broken-line plane. But on average, there will be just as much molecules crossing the plane from below as there will be crossing the plane from above. So the net numbers of molecules above and below the plane do not change. That is illustrated in the figure by a pair of molecules crossing the plane from opposite sides.

Now if the velocity profile was uniform, the x-momentum crossing the plane from below would also cancel the x-momentum crossing the plane from above. So there would be no net x-momentum transfer through the plane either. But if the velocity profile is sloped as in the figure, the molecule coming from above comes from a region of somewhat higher average velocity. Higher by an amount of the order of $\lambda \partial u/\partial y$ in fact. Molecules like that will on average take this excess momentum out of the fluid above the broken line and dump it into the fluid below the broken line. So the fluid above the line slows down and the fluid below the line speeds up. Molecules crossing the plane from below have the same effect: they take a lack of momentum out of the lower fluid, speeding it up, and dump it in the upper fluid, slowing it down.

It is as if the lower fluid exerts a shear force on the upper fluid that slows it down, with an opposite reaction force where the upper fluid speeds up the lower fluid. How big is that shear force? Well, the mass of molecules crossing the layer from either side per unit area and unit time is of order ρa. (Really less: only half of the molecules near the plane are going towards it, and only the vertical component of the random motion helps them get to the plane. But we are only ballparking things here.) The excess or lack of velocity being transfered across the plane is of order $\lambda \partial u/\partial y$. Since force is rate of momentum change, and momentum is mass times velocity, the viscous stress per unit area is therefore $\rho a \lambda \partial u/\partial y$.

That makes the kinematic viscosity ν of order $a\lambda$. You could also have concluded that from a simple dimensional analysis. But now you have the physical idea that can be extended to turbulence.

Exercise:

Compare the value of λa of standard air with its dynamic viscosity.

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4 Mixing length

(Panton3 26.4)

The "mixing length" idea tries to model the turbulent Reynolds stress much like the molecular motion created the laminar stress in the previous section. The random motion of coherent regions, "eddies," of the turbulent fluid is assumed to be equivalent to the random motion of the molecules in the laminar case. As an equivalent to the free path length between molecular collisions, it is assumed that there is some "mixing length" ℓ over which the eddies exchange momentum. Fluid at a given plane originates from some transverse distance ℓ away, with much of its velocity difference left intact during the trip.

Assuming all of that is true, the kinematic eddy viscosity would be $\ell v'$, where v' is the typical vertical velocity fluctuation. In other words, $\lambda \to \ell$ and $a \to v'$. From continuity, $u'_x + v'_y = 0$, so assuming that there is no

strong directionality in eddy length scales, v' is of the order u'. And u' was estimated implicitly as $\ell \partial u/\partial y$ in the previous section, so the magnitude of that must be the estimate for the random transverse velocity v'. That gives a turbulent eddy viscosity

$$\boxed{\nu_{\mathrm{T}} = \ell^2 \left| \frac{\partial \overline{u}}{\partial y} \right|}$$

Assuming ℓ is a known quantity, this can be plugged into the averaged equations. Then they can be solved on some computer, since the Reynolds stresses are no longer unknowns. The estimate of ℓ could be the typical transverse length scale in free turbulence or the distance from the wall in a surface layer.

Unfortunately, many things are wrong in the story. For one, turbulent eddies are not small compared to the transverse scales of the flow, so $u' \neq \ell \partial u/\partial y$. The value of u' is in fact related to the velocity at finite distances. This makes the entire idea of having universal partial differential equations (involving local derivative quantities only) unsound. (Even though many people seem to believe this fundamental problem will somehow go away if you make the local derivative terms complex enough, like in "second order modelling.") Also, at least the larger turbulence scales are definitely directional. And the turbulent fluctuations are not independent of the mean velocity field like a is.

Worse, the turbulent shear stress would always be predicted to be exactly zero at points of $\partial \overline{u}/\partial y = 0$ even if there is no flow symmetry around that point, which is obviously nonsense. And surely, even if there is symmetry, the eddy viscosity should not be zero at the symmetry line. It is often a better idea to replace $\ell |\partial \overline{u}/\partial y|$ by some typical turbulent velocity. That gives a constant eddy viscosity.

Still, the mixing length idea is qualitively perhaps one of the most useful tools available because it is so simple. The more complex you make a turbulence model, the more things can go wrong. A mixing length model may be very inaccurate, but it does tend to show the correct general ideas.

Also, the mixing length idea becomes more believable if you solve the larger turbulence scales on a computer, and just model the smaller velocity fluctuations as an effective viscosity. That idea is called "large eddy simulation." Even then, it is not a trivial modelling problem, unfortunately. Unlike the laminar case, there is no separation of scales between the scales you describe and the ones you model.

5 Energy cascade

(Panton3 26.9,10)

In trying to understand turbulence, it helps to have some mental picture of it. Suppose you look at a typical turbulent flow in some boundary layer, jet, mixing layer, or whatever. You will perceive organized masses of fluid, "eddies," in random motion, distorting while moving. The size of these eddies will be quite comparable to the thickness of the turbulent layer or jet. But if you look closer, you see that there are also smaller scale fluctuations, smaller eddies, that seem to do their own independent thing. Nonlinear motion on larger scales tends to create motion on smaller scales. (Much like squaring a $\cos x$ produces a $\cos 2x$ with half the wave length.) The idea here is that the larger eddies put some of their kinetic energy in creating smaller eddies, which in turn create still smaller eddies, and so on.

Now the motion of the smaller eddies involves less velocity fluctuations relative to their surroundings. Therefore the largest eddies have most of the turbulent kinetic energy per unit mass

$$\frac{1}{2}\sqrt{\overline{u_i' u_i'}}$$

Similarly, the largest eddies will dominate the Reynolds stresses

$$\tau_{ij}^{\mathrm{R}} = \rho \overline{u_i' u_j'}$$

experienced by the mean flow. However, the smallest eddies, because they are small, will dominate the turbulent velocity *derivatives* and with it the dissipation per unit mass

$$\frac{\varepsilon}{\rho} = 2\nu s_{ij} s_{ij} \qquad s_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

that turns their kinetic energy irreversibly into heat. The picture is that there is an "energy cascade" where the largest eddies put kinetic energy into smaller ones, these smaller ones into still smaller ones, until extremely small final eddies convert that kinetic energy into heat. The typical scales of these final smallest eddies are called the Kolmogorov scales. For high Reynolds number, (small ν), the difference between the largest and smallest scales can be tremendous.

To see that, consider first what governs the smallest eddies. One important factor is of course the kinetic energy ϵ/ρ that is draining through the cascade to the smallest eddies, for them to dissipate it. The other important factor is of course the kinematic viscosity ν that allows them to dissipate it in the first place. The eddies are presumably too small to "see" the large scale features of the flow, so ϵ/ρ and ν are the only two quantities that should govern the small eddies. Let's do some dimensional analysis based on that. If $\eta_{\rm K}$ is the typical length scale of the smallest eddies, $\tau_{\rm K}$ the typical time scale, and $\upsilon_{\rm K}$ the typical velocity, the corresponding three nondimensionsl Π groups that you can form are, noting that the viscosity ν has units L^2/T and the dissipation per unit mass ε/ρ units $\mathrm{L}^2/\mathrm{T}^3$:

$$\eta_{\rm K}\sqrt[4]{\frac{\varepsilon}{\rho\nu^3}} \qquad \tau_{\rm K}\sqrt{\frac{\varepsilon}{\rho\nu}} \qquad \upsilon_{\rm K}\sqrt[4]{\frac{\varepsilon\nu}{\rho}}$$

Since these Π groups have nothing else to depend on, they must be finite numbers. In fact, we can *define* specific and meaningful scales of the smallest eddies by setting each Π group equal to 1. The thus defined $\eta_{\rm K}$, $\tau_{\rm K}$, and $\upsilon_{\rm K}$ are called the Kolmogorov microscales:

$$\boxed{\eta_{\rm K} = \sqrt[4]{\frac{\rho\nu^3}{\varepsilon}} \qquad \tau_{\rm K} = \sqrt{\frac{\rho\nu}{\varepsilon}} \qquad \upsilon_{\rm K} = \sqrt[4]{\frac{\rho}{\varepsilon\nu}}}$$

To get a general idea how big those Kolmogorov scales really are, we will have to estimate ε/ρ, the kinetic energy dissipated into heat per unit time and unit mass. Well, in a steady state, the amount of energy that is dissipated by the small eddies must cancel the amount of energy that the largest eddies put into the cascade. Let's try to estimate the latter. The amount of kinetic energy put into the cascade by the largest scale eddies per unit time and unit mass should presumably be proportional to the typical kinetic energy of the largest scale eddies, call it $\frac{1}{2}v_{\rm L}^2$, with $v_{\rm L}$ a typical turbulent velocity of the largest eddies, and inversely proportional to the time it takes the eddies to evolve nontrivially, estimated as $v_{\rm L}/\ell$ where ℓ is the typical size of the largest eddies. So the kinetic energy put into the cascade is estimated to be of order $v_{\rm L}^3/\ell$. If you put that into the unit Kolmogorov Π groups above, you find the following ratios of the smallest to the largest eddy scales:

$$\boxed{\frac{\eta_{\rm K}}{\ell} = Re^{-3/4} \quad \frac{\tau_{\rm K}}{\ell/v_{\rm L}} = Re^{-1/2} \quad \frac{\upsilon_{\rm K}}{v_{\rm L}} = Re^{-1/4} \qquad Re \equiv \frac{v_{\rm L}\ell}{\nu}}$$

From this it follows that for large Reynolds number Re, the smallest eddies can be much, much, smaller than the largest ones. Note further that normally the turbulent velocity $v_{\rm L}$ of the largest eddies will be similar to the total turbulent velocity, since the largest eddies have most of the turbulent kinetic energy. And the total turbulent velocities in turn are typically quite comparable to the average flow velocity. Also the size ℓ of the largest eddies is typically quite comparable to the transverse thickness of the vortex layer or jet. So the Reynolds number above can be taken to be roughly one based on flow velocity and layer or jet thickness.

How about if you plot the kinetic energy in the entire cascade? In particular, assume that you take a Fourier transform and then plot the total kinetic energy per unit mass and unit wave number range P versus wave number k? That is called a power spectrum. (It is a consequence of the orthogonality of Fourier modes

that you can write the kinetic energy as a simple sum of the contributions of each individual mode. For Fourier modes, this is called "Parseval's identity.") Assume now that you look at the power spectrum for wave numbers that are so small that the large scale features of the turbulent flow are no longer visible to the eddies, but not so small that dissipation becomes a factor. In that range of wave lengths, called the "inertial range," the viscosity ν is not important and the power spectrum can only depend on ε/ρ. That has units of $\mathrm{L}^2/\mathrm{T}^3$, while the wave number k has units $1/\mathrm{L}$ and the power spectrum P has units $\mathrm{L}^3/\mathrm{T}^2$. The only Π group you can form here is

$$\frac{Pk^{5/3}}{(\varepsilon/\rho)^{2/3}}$$

This Π group has nothing it can depend on, so it must be a finite constant. That means that P must be proportional to $k^{-5/3}$ in the inertial range. If you plot P versus k using logarithmic axes, the inertial range will be seen to be a straight line segment that slopes down with slope $-5/3$.

6 Free Turbulence

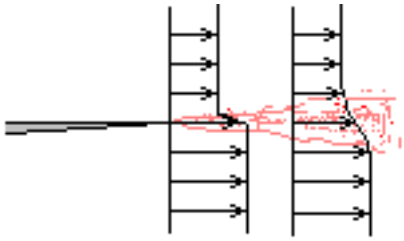
(Panton3 23.5, 6, Tennekes & Lumley 4)

Maybe someday an Oracle of Delphi will arise that can actually model turbulence accurately. Do not hold your breath.

Until then, the most solid way to learn some real stuff about turbulence is without doubt dimensional analysis. Dimensional analysis does not require you to actually solve the equations. It just requires you to figure out what is important and what is not.

Some important examples follow.

Mixing layers:



Consider a turbulent mixing layer (AKA shear or vortex layer) as shown above. Going downstream, you would expect the velocity profile to be of the generic functional form

$$\overline{u} = F(x, y, U_{\mathrm{ave}}, \Delta U, \nu, \text{initial conditions})$$

where x is the horizontal distance from the starting point of the layer, y the vertical coordinate through the layer, U_{ave} the average of the velocities above and below the layer, and ΔU the velocity change over the layer.

However, in a free turbulent flow, away from walls, the laminar shear stress is small compared to the turbulent one. This suggests that the laminar viscosity ν can be ignored in the relation above. (Remember from the energy cascade discussion that the laminar stress is important in the dissipation of turbulent kinetic energy. But it should not be involved in the large-scale mechanics of the turbulent mixing layer, like the instability mechanisms that sustain the big eddies. If ν changes, it should only affect at which scales the energy cascade gets rid of its energy.)

Also, if you look sufficiently far downstream, the details of the initial conditions should no longer be visible.

Dimensional analysis can now be done for the mean velocity profile $\overline{u}$ above. Selecting x and ΔU to nondimensionalize the remaining variables gives

$$\boxed{\text{Mixing layer:} \quad \frac{\overline{u}}{\Delta U} = f\left(\frac{y}{x}, \frac{U_{\text{ave}}}{\Delta U}\right)}$$

But the second argument of f above is a constant, independent of x for a given mixing layer. So it follows that the mixing layer velocity profile is similar, and that the layer has a typical thickness δ proportional to x.

Exercise:

Discuss how well that seems to agree with flow visualizations.

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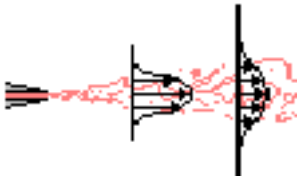
Note that while in a boundary layer approximation the mixing layer can reasonably be approximated as relatively thin compared to its streamwise extend, this approximation does not improve with streamwise distance.

Jets:



Now consider a turbulent jet, as shown above. Here you would expect that downstream

$$\overline{u} = F(x, y, \nu, \text{initial conditions})$$

Again, ν can be ignored, assuming that the turbulent Reynolds number is large enough.

Also, it can again be assumed that the details of the initial conditions become invisible sufficiently far downstream, with one exception. Integral momentum conservation between any two downstream positions of constant x implies that the x-momentum flow integral

$$\int \rho \overline{u}^2 \, \mathrm{d}A$$

must be the same at the two stations. (Pressure differences between stations sufficiently far downstream can be ignored). So the momentum flow integral above is a constant. It is determined by the strength of the jet that the initial conditions generated. Since it is constant, it *cannot* become invisible. Note that in the incompressible case, you can more simply assume that

$$I_0 \equiv \int \overline{u}^2 \, \mathrm{d}A$$

is constant. Also note that I_0 has units $\mathrm{L}^3/\mathrm{T}^2$ for a two dimensional jet, where $\mathrm{d}A = \mathrm{d}y$, but units $\mathrm{L}^4/\mathrm{T}^2$ for a three-dimensional jet, for which $\mathrm{d}A = r \, \mathrm{d}r \, \mathrm{d}\theta$.

So the functional dependence can be simplified to

$$\overline{u} = F(x, y, I_0)$$

and dimensional analysis then produces

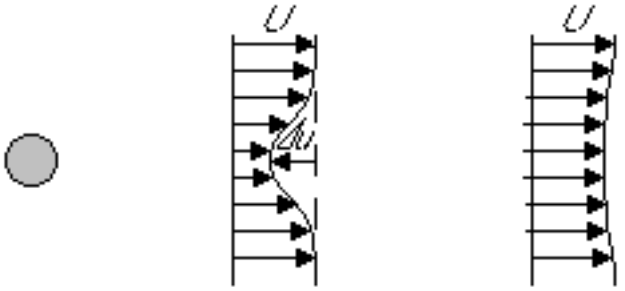
$$\text{2D: } \frac{\overline{u}\sqrt{x}}{\sqrt{I_0}} = f\left(\frac{y}{x}\right) \qquad \text{3D: } \frac{\overline{u}x}{\sqrt{I_0}} = f\left(\frac{y}{x}\right)$$

Cleaning this up gives:

$$\boxed{\text{2D jet: } \quad \overline{u} = \frac{\sqrt{I_0}}{\sqrt{x}} f\left(\frac{y}{x}\right) \qquad \text{3D jet: } \quad \overline{u} = \frac{\sqrt{I_0}}{x} f\left(\frac{y}{x}\right)}$$

Like for the mixing layer, the mean velocity profiles are similar, and the jet thickness is proportional to x. But in the two-dimensional case, the maximum jet velocity decays proportional to $1/\sqrt{x}$, slower than the $1/x$ of the three-dimensional case.

Wakes:



Finally consider the wake of some body, as shown above. Here you would expect that downstream

$$\Delta u = F(x, y, U, \nu, \text{initial conditions}) \qquad \text{where } \Delta u \equiv U - \overline{u}$$

Again, ν can be ignored. Also, it can again be assumed that the details of the initial conditions become invisible sufficiently far downstream, with the exception that

$$I_0 \equiv \int \overline{u}^2 \, \mathrm{d}A = \int (U - \Delta u)^2 \, \mathrm{d}A$$

cannot become invisible because it is constant. If Δu has become small enough, we can expand the square and ignore the $(\Delta u)^2$ term to give that

$$\int U^2 \, \mathrm{d}A - 2 \int U \Delta u \, \mathrm{d}A$$

is constant. Since the first term is just a constant too, more simply

$$I_1 = \int \Delta u \, \mathrm{d}A$$

must be constant. In two dimensions this has units L^2/T and in three L^3/T.

The functional relationship simplifies to

$$\Delta u = F(x, y, U, I_1)$$

and dimensional analysis then gives

$$\text{2D: } \frac{\Delta u x}{I_1} = f_2\left(\frac{y}{x}, \frac{Ux}{I_1}\right) \qquad \text{3D: } \frac{\Delta u x^2}{I_1} = f_2\left(\frac{y}{x}, \frac{Ux^2}{I_1}\right)$$

Because of the second argument of function f_2, there is little useful knowledge that we can get from this.

However, we can supplement the dimensional analysis with what we believe to be true about the turbulence. Consider the two-dimensional Reynolds-averaged x-momentum equation in boundary layer approximation:

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = \frac{1}{\rho}\frac{\partial \overline{\tau_{xy}} + \tau_{xy}^{\mathrm{R}}}{\partial y}$$

The average laminar stress should be negligible and when Δu has become small enough compared to U, we can also approximate the left hand side to give

$$-U\frac{\partial \Delta u}{\partial x} = \frac{1}{\rho}\frac{\partial \tau_{xy}^{\mathrm{R}}}{\partial y} \qquad \frac{\tau_{xy}^{\mathrm{R}}}{\rho} = \overline{u'v'}$$

(Note that $\overline{v}\partial/\partial y$ should, based on continuity, be of the same order as $\overline{\Delta u}\partial/\partial x$, hence negligible compared to the retained term.)

If we ballpark the two sides in the simplified momentum equation above,

$$\frac{U\Delta u}{x} \sim \frac{(\Delta u)^2}{\delta}$$

or rearranged

$$\frac{\Delta u}{\delta} \sim \frac{U}{x}$$

In particular, there would be a problem in reasonably balancing the momentum equation if $\Delta u/\delta$ would be proportional to a different power of x for large x than x^{-1}. In addition, since I_1 is constant, in two dimensions we also have the constraint that $\Delta u\delta$ must stay of order x^0. Combining the two, in two dimensions, as far as powers of x are concerned, we must have that

$$\Delta u \sim \frac{1}{\sqrt{x}} \qquad \delta \sim \sqrt{x}$$

Now we trivially rewrite the previous relationship for Δu as

$$\frac{\Delta u x}{I_1}\sqrt{\frac{I_1}{Ux}} = \sqrt{\frac{I_1}{Ux}} f_2\left(\frac{y}{x}\sqrt{\frac{Ux}{I_1}}\sqrt{\frac{I_1}{Ux}}, \frac{Ux}{I_1}\right)$$

Here we have cleverly formed nondimensional combinations for Δu and y that should be independent of x for large x. But the right hand side can be written as just a different function f:

$$\frac{\Delta u x}{I_1}\sqrt{\frac{I_1}{Ux}} == f\left(\frac{y}{x}\sqrt{\frac{Ux}{I_1}}, \frac{Ux}{I_1}\right)$$

Function f is not a new function, it is fully determined by f_2: if you know the arguments of f, you can compute those of f_2 and then f. But function f cannot depend on its second argument, or else the shown nondimensional combinations for Δu and y would not be independent of x as they should.

Cleaning up then gives

$$\boxed{\text{2D wake:} \quad \frac{\Delta u}{U} = \sqrt{\frac{I_1}{Ux}} f\left(\frac{y}{x}\sqrt{\frac{Ux}{I_1}}\right)}$$

It may be noted that if substitute the above similar profile into the simplified momentum equation above, assuming some suitable constant eddy viscosity ν_{T} for the viscous term, you can find the profile. You will get a reasonable approximation to the actual measured wake velocity profile except near the outer edges. Note that near the outer edges the flow is only part of the time truly turbulent, as near the outer edges turbulent eddies engulf regions of potential flow fluid. The "intermittency" γ is defined as the fraction of time that the

flow is turbulent. It turns out that if you assume that $\nu_T = \nu_{T,center}\gamma$, with $\nu_{T,center}$ a suitable constant, you can get very good agreement with the experimental profile.

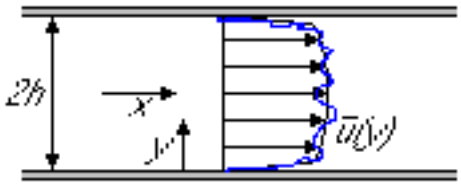
In three dimensions, accounting for the different I_1,

$$\boxed{\text{3D wake:} \quad \frac{\Delta u}{U} = \sqrt[3]{\frac{I_1}{Ux^2}} f\left(\frac{y}{x}\sqrt[3]{\frac{Ux^2}{I_1}}\right)}$$

Note that in this case, the Reynolds number based on Δu and δ is proportional to $x^{-1/3}$, so it decreases with x. Therefore the made assumption of high Reynolds number will eventually break down, and the wake will even become laminar. The result above assumes that you do not look that far downstream. (The two-dimensional wake above and the three-dimensional jet have Reynolds numbers that stay constant with x, so the approximation of high Reynolds number, while qualitatively quite reasonable, does not improve with x in those cases.)

7 Wall-bounded flows

Channel Flow



A typical example of a wall-bounded flow is turbulent duct flow, as sketched above.

Assuming a long duct, the velocity-related quantities and the pressure gradients will be independent of x, just like for the laminar case. That makes turbulent duct flow simpler than turbulent boundary layers, where the streamwise gradients are smaller than the transverse ones, but not zero. Consider how the Reynolds-averaged Navier-Stokes equations simplify for the duct:

Continuity

$$\frac{\partial \overline{v}}{\partial y} = 0 \quad \Longrightarrow \quad \overline{v} = 0$$

x Momentum:

$$0 = -\frac{\partial \overline{p}}{\partial x} + \frac{\partial \tau + \tau^R}{\partial y} \qquad \tau \equiv \overline{\tau_{xy}} = \mu \frac{\partial \overline{u}}{\partial y} \quad \tau^R \equiv \tau^R_{xy} = -\rho \overline{u'v'}$$

y-Momentum:

$$0 = -\frac{\partial \overline{p}}{\partial y} + \frac{\partial \overline{\tau^R_{yy}}}{\partial y} \qquad \tau^R_{yy} = -\rho \overline{v'v'}$$

The y-momentum equation can easily be integrated to give

$$\boxed{p + \rho \overline{v'v'} = p_w(x)}$$

where p_w is the pressure on the walls. So, unlike in laminar duct flow, the pressure is not independent of y; it is highest at the walls, (where $v' = 0$ because of the wall boundary conditions), and lowest in the center of the duct.

If we put the above pressure into the x-momentum equation, noting that $\rho\overline{v'v'}$ as a velocity quantity is independent of x, we get:

$$\boxed{\tau + \tau^{\mathrm{R}} = -\frac{\mathrm{d}p_w}{\mathrm{d}x}(h-y) \quad \tau = \mu\frac{\partial\overline{u}}{\partial y} \quad \tau^{\mathrm{R}} = -\rho\overline{u'v'}}$$

where y is taken to be zero on the center line of the duct. So, the *total* stress, laminar plus Reynolds, is linear in y, and vanishes (by symmetry) on the center line.

Now a turbulent flow will dissipate much more kinetic energy than a laminar one under the same conditions. The irreversible loss of kinetic energy must come from a much larger pressure gradient pushing the flow through the duct. But since the net force on the fluid must be zero, the shear force exerted by the wall on the fluid must be much greater too. In short, in the above equation, both the pressure gradient and the *net* shear stress are much larger than in the laminar case.

Now at the walls, the Reynold stress $\tau^{\mathrm{R}} = \rho\overline{u'v'}$ is zero because of the wall boundary condition on the velocity. So there must be a very large *laminar* stress τ at the walls. And since $\tau = \mu\partial\overline{u}/\partial y$, the velocity profile must be varying extremely rapidly near the walls. But, since the profile is monotonous to the center, this can only be true in a thin layer, called the "surface layer," at each of the two walls. Outside these layers, the Reynolds stress must dominate and the laminar stress can be ignored.

Exercise:

> Sketch the velocity profiles for laminar and turbulent flow with the same mass flow through the duct. Indicate the layer near the wall and how it is different from the laminar flow case.

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By convention, the magnitude of the shear stress at the wall is written in terms of a "friction velocity" u^*:

$$\boxed{\tau_0 = -h\frac{\mathrm{d}p_w}{\mathrm{d}x} \equiv \rho {u^*}^2}$$

It is a measure of how much stress the overall flow exerts on the fluid near the wall.

It seems reasonable to assume that in the thin surface layers, the large-scale features of the flow are only evident through the stress they impose, as measured by u^*. So if we take Δy to be the distance from the wall, we can assume that

$$\overline{u} = f_0(\Delta y, u^*, \nu)$$

Then dimensional analysis gives the "law of the wall":

$$\boxed{\frac{\overline{u}}{u^*} = f\left(\frac{u^*\Delta y}{\nu}\right)}$$

This says that in the surface layer, the nondimensional flow depends only on an effective Reynolds number formed with the distance from the wall. This Reynolds number decreases to zero when approaching the wall. In fact, close to the wall there is a sublayer in which the Reynolds number becomes too low for turbulence to sustain itself. This sublayer is called the "viscous sublayer."

In the center region of the duct, in between the two surface layers, the picture to keep in mind is that the strong turbulent mixing would create a uniform velocity profile if there were no walls. However, the surface layers exert a shear force, measured by u^*, on the edges of the center region, preventing the uniform profile

to fully develop. It seems reasonable then to assume that the deviation from a uniform profile is functionally given by

$$\overline{u} - u_{\max} = F_0(y, h, u^*)$$

Dimensional analysis then produces the so-called "velocity defect law":

$$\boxed{\frac{\overline{u} - u_{\max}}{u^*} = F\left(\frac{y}{h}\right)}$$

Now consider some hypothetical thin layer, close to the bottom wall, but still well above the still thinner surface layer. In such a layer, both the law of the wall (for $\Delta y u^*/\nu$ relatively large) *and* the velocity defect law (for $y \approx -h$) must be valid. Such a layer, in which two different expressions are both valid, is called a "matching layer." For turbulent boundary layers, the matching region is referred to as the "inertial sublayer." (But "inertial region" would have been a more accurate name since a matching region has no clearly defined boundaries. Note also that by definition the inertial sublayer has to be thick compared to the surface layer below it and thin compared to the duct size.)

If both the law of the wall and the velocity defect law must be valid in the inertial sublayer, then their y derivatives must match too. Writing that out gives:

$$\frac{y}{u^*}\frac{\mathrm{d}u}{\mathrm{d}y} = f'\left(\frac{yu^*}{\nu}\right)\frac{yu^*}{\nu} = F'\left(\frac{y}{h}\right)\frac{y}{h}$$

And if two functions of different arguments must be the same, then each must be a constant. (Otherwise you could simply change just one of the arguments to create inequality.) So the two expressions above must be a constant. This constant is normally indicated as $1/\kappa$, where κ is called the "von Kármán's" constant. Its value is often quoted as 0.4, but it might be somewhat less, like 0.33, at high enough Reynolds numbers.

The two expressions above can now be integrated to give the shape of the velocity profile in the inertial sublayer:

$$\boxed{\frac{\overline{u}}{u^*} = \frac{1}{\kappa}\ln\left(\frac{yu^*}{\nu}\right) + C_f \qquad \left(\frac{yu^*}{\nu} \to \infty\right)}$$

$$\boxed{\frac{\overline{u} - u_{\max}}{u^*} = \frac{1}{\kappa}\ln\left(\frac{y}{h}\right) + C_F \qquad \left(\frac{y}{h} \to 0\right)}$$

The derivation of this logarithmic profile by Millikan in 1939 was one of the landmark events in turbulence theory.

Substracting the two equations above, we get the "logarithmic friction law":

$$\boxed{\frac{u_{\max}}{u^*} = \frac{1}{\kappa}\ln\left(\frac{hu^*}{\nu}\right) + C}$$

Recall that u^* is really a measure of the wall shear, or equivalently the pressure gradient needed to keep the flow moving. So the above equation relates the pressure gradient needed to drive the flow to the maximum velocity in the duct.

Exercise:

Discuss figure 26.13 and how it verifies and does not verify the above discussion for the case of a boundary layer.

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