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Abstract

In order to simulate diffusion processes in discrete vortex computations, the vorticity of each vortex is redistributed over its neighbors. A sufficient condition for convergence is derived. This condition is cast as the solution of an underdetermined system of equations. Computed examples of diffusion in one space dimension are compared to random–walk results.

1. Introduction.

Vortex methods show promise for incompressible flow at high Reynolds numbers since only the vorticity must be described, saving storage. In addition, Van Dommelen & Rundensteiner [1] introduced a fast numerical scheme in which the number of numerical operations per vortex is finite, so that flows with limited regions of vorticity can be computed rapidly. In a vortex method, the computational domain can truly be infinite.

Besides, when combined with, for example, a random walk simulation of the diffusion terms in the Navier–Stokes equations, the method can be completely mesh-free. This makes it possible to address complex multi-component configurations, such as a missile which is fired from a submarine, in a simple way.

Yet the random walk method has several significant disadvantages, such as random perturbations in the results that introduce an awkward uncertainty in interpretation. In addition, the center of vorticity is not preserved, and can drift sizably for longer times. Such a drift can have sizable influence on the flow. Finally, the asymptotic order of accuracy is low.

This paper examines an alternate mesh-free algorithm which achieves the diffusion through a process of redistribution of vortex strength. During each time-step, the circulation of each vortex is redistributed over adjacent vortices in a way to give the correct amount of diffusion. Such a procedure is consistent with the fast velocity summation of [1] that already groups neighboring vortices together to reduce the number of numerical operations required.

This method is deterministic; its convergence for the Stokes' equations is proved in section 3. A 'stability' condition must be met that can be cast as a standard problem involving the solution of an underdetermined system of linear equations. The solution of that problem is fairly easy; it is roughly equivalent to the generation of suitable stable finite difference formulae for an arbitrary set of points. In addition, the accuracy can in principle be as high as desired, when a sufficient vortex density exists, simply by increasing the number of equations in the linear system. The method preserves the center of vorticity of each vortex exactly.

Some preliminary simple one-dimensional numerical experiments are presented in section 4. While the method seems to generalize straightforwardly to the two-dimensional Navier-Stokes equations, the question whether the vortex stretching in three-dimensional flow can be handled by a similar redistribution procedure must be addressed separately.

2. Basic Method.

In the vortex redistribution method, the vorticity is at each time step $j = 0, 1, \ldots$

described as a set of vortex blobs;

$$\omega_j = \sum_p \gamma_p \frac{1}{\ell^2} \phi \left(\frac{|\mathbf{r} - \mathbf{r}_p|}{\ell} \right) \tag{2.1}$$

where γ_p is the circulation of vortex p and ϕ is a symmetric smoothing function that integrates to unity. For convenience, we will assume that the number of dimensions is two, although this is not essential. Given a more general initial vorticity distribution than (2.1), it may be convolved with the function ϕ and the convolution approximated by the trapezium rule to find an approximation of the form (2.1).

As usual in vortex procedures, the convection terms in the Navier-Stokes equations are modelled through the motion of the vortices. The purpose of the redistribution algorithm is to approximate the viscous diffusion during each time step Δt. This is done by redistributing the strength γ_p of each vortex p over vortices q in a neighborhood of vortex p. Thus the new vorticity distribution assumes the form

$$\omega_{j+1} = \sum_{p,q} \gamma_p w_{pq} \frac{1}{\ell^2} \phi \left(\frac{|\mathbf{r} - \mathbf{r}_q|}{\ell} \right) \tag{2.2}$$

where w_{pq} is the relative fraction of γ_p assigned to vortex q.

Since the vortex positions are assumed to be given by the convection approximation, the correct diffusion effects must be obtained by a suitable choice of the redistribution weights w_{pq}. Conditions for these weights follow from taking the Fourier transform of the vorticity distribution,

$$\hat{\omega}_j = \hat{\phi}(k\ell) \sum_p \gamma_p e^{-i\mathbf{k}\cdot\mathbf{r}_p} \tag{2.3}$$

where $\mathbf{k}$ is the wave number vector. Similarly, the redistributed vorticity distribution transforms as

$$\hat{\omega}_{j+1} = \hat{\phi}(k\ell) \sum_{p,q} \gamma_p w_{pq} e^{-i\mathbf{k}\cdot\mathbf{r}_q} \tag{2.4}$$

On the other hand, after time step Δt exact diffusion would give the transformed vorticity

$$\hat{\omega}^e_{j+1} = \hat{\phi}(k\ell) \sum_p \gamma_p e^{-i\mathbf{k}\cdot\mathbf{r}_p} e^{-k^2\nu\Delta t} \tag{2.5}$$

where k is the magnitude of $\mathbf{k}$.

The general procedure is now to choose the redistribution fractions w_{pq} to bring the expression (2.4) in agreement with the exact result (2.5) as long as the k is finite. To this end, (2.4) is rewritten to a form similar to (2.5),

$$\hat{\omega}_{j+1} = \hat{\phi}(k\ell) \sum_p \gamma_p e^{-i\mathbf{k}\cdot\mathbf{r}_p} \sum_q w_{pq} e^{-i\mathbf{k}\cdot\mathbf{d}_{pq}\sqrt{2\nu\Delta t}} \tag{2.6}$$

where

$$\mathbf{d}_{pq} = \frac{\mathbf{r}_q - \mathbf{r}_p}{\sqrt{2\nu\Delta t}} \tag{2.7}$$

Expanding both (2.5) and (2.6) into a Taylor series for small Δt, the leading order equations become:

$$\sum_q w_{pq} = 1 \tag{2.8}$$

$$\sum_q w_{pq} d_{1pq} = 0 \qquad \sum_q w_{pq} d_{2pq} = 0 \tag{2.9b,c}$$

$$\sum_q w_{pq} \left(d_{1pq}\right)^2 = 1 \qquad \sum_q w_{pq} d_{1pq} d_{2pq} = 0 \qquad \sum_q w_{pq} \left(d_{2pq}\right)^2 = 1 \tag{2.10a,b,c}$$

$$\sum_q w_{pq} \left(d_{1pq}\right)^3 = 0 \qquad \sum_q w_{pq} \left(d_{1pq}\right)^2 d_{2pq} = 0 \tag{2.11a,b}$$

$$\sum_q w_{pq} d_{1pq} \left(d_{2pq}\right)^2 = 0 \qquad \sum_q w_{pq} \left(d_{2pq}\right)^3 = 0 \tag{2.11c,d}$$

Thus a system of linear equations for the unknown weights w_{pq} results.

Equation (2.8) expresses the preservation of the circulation of each vortex, (2.9) the conservation of the center of vorticity, and (2.10) the leading order diffusion. This can be compared with the random walk procedure, which does preserve circulation, but preserves the center of vorticity only in the limit of infinitely many vortices.

For consistency, at least equations (2.8) through (2.10) must be satisfied, while further equations such as (2.11) may be included to improve the accuracy. There is no fundamental limitation on the order of accuracy that can be achieved, although the system of equations may become ill-conditioned for higher accuracy. And obviously, the number of vortices q in the neighborhood of any vortex p must at least equal the number of equations retained in (2.8), (2.9), (2.10),... Since the inclusion of vortices at distances much larger than $\sqrt{2\nu\Delta t}$ decreases accuracy (cf. the next section), a sufficient density of vortices must be present. If it is not, new vortices of zero strength can be added.

There is an additional restriction on the solution of (2.8) and following; the convergence proof of the next section requires that all weights w_{pq} are non-negative. If the initial vortices are confined to a finite domain, the non-negativity condition cannot be satisfied at the boundaries of that domain, and new vortices must be added at each time step. Physically this addition of points allows the vorticity to spread over the total flow region, instead of being confined to the initial domain.

The main problem is to find a non-negative solution to (2.8) and following, given a neighborhood of points $\mathbf{r}_q$. Note that we will want to include sufficient points to make the existence of a non-negative solution w_{pq} likely, in which case the system of equations will

usually be underdetermined. From all possible solutions, we need to select a non-negative one, if it exists. To do that, we first note that on behalf of (2.8) all weights w_{pq} must be less than unity, so that we need a solution in the range

$$0 \leq w_{pq} < 1$$

Alternatively, if we introduce shifted unknowns

$$u_q = w_{pq} - \tfrac{1}{2} \tag{2.12}$$

we need a solution

$$\max(|u_q|) \leq \tfrac{1}{2} \tag{2.13}$$

Therefor, if we find the solution with the least possible maximum, it will be an acceptable solution if the maximum is less than $\frac{1}{2}$. If it is greater, the set of vortices is insufficient and one or more new vortices must be added. An efficient algorithm to find the least maximum solution to an underdetermined system is described in [2].

3. Convergence For The Stokes Equations

In this section, we will examine the convergence of the redistribution algorithm. Since the convection terms are described separately as vortex motion, in the analysis we will restrict ourselves to the solution of the Stokes equations. We will first estimate the difference

$$\delta_j = \hat{\omega}_{j+1} - \hat{\omega}_j e^{-k^2 \nu \Delta t} \tag{3.1}$$

between the redistributed solution (2.6) and the exactly diffused solution (2.5).

By satisfying (2.8) and following, the first few terms in the Taylor series expansions with respect to time for the redistributed and exact solutions agree. If $M \geq 3$ is the order of the first term in the Taylor series expansion of the redistributed solution that is not equated to a corresponding term in the exactly diffused solution, then the Taylor series remainder theorem gives:

$$|\delta_j| = \left| \hat{\phi}(k\ell) \sum_p \gamma_p e^{-i\mathbf{k}\cdot\mathbf{r}_p} \left(\left\{ \sum_q w_{pq} \frac{(-1)^M i^M}{M!} \sqrt{2\nu\Delta t}^M (\mathbf{k}\cdot\mathbf{d}_{pq})^M [\cos(\alpha_1) + i\sin(\alpha_2)] \right\} \right.$$
$$\left. - \frac{(-1)^N}{2^N N!} \sqrt{2\nu\Delta t}^{2N} k^{2N} e^{-\alpha k^2 \nu \Delta t} \right) \Bigg| \tag{3.2}$$

where $N = M/2$ when M is even or $N = (M+1)/2$ when M is odd, while $0 \leq \alpha \leq 1$. Using the triangle equality,

$$|\delta_j| \leq |\hat{\phi}(k\ell)| \sum_p |\gamma_p| \left(\sum_q w_{pq} \frac{1}{M!} \sqrt{2\nu\Delta t}^M |\mathbf{k}\cdot\mathbf{d}_{pq}|^M \sqrt{2} + \frac{1}{2^N N!} \sqrt{2\nu\Delta t}^{2N} k^{2N} \right) \tag{3.3}$$

where we used the fact that the w_{pq} are non-negative.

If the maximum allowed distance between any point p and any neighboring point q is d_0,

$$|\mathbf{d}_{pq}| \leq d_0 \tag{3.4}$$

the sum in (3.3) can be further bounded using (2.10),

$$\sum_q w_{pq} |\mathbf{k} \cdot \mathbf{d}_{pq}|^M \leq \max\left(|\mathbf{k} \cdot \mathbf{d}_{pq}|^{M-2}\right) \sum_q w_{pq} (\mathbf{k} \cdot \mathbf{d}_{pq})^2 \leq k^M {d_0}^{M-2} \tag{3.5}$$

Further, the total absolute circulation may be defined as

$$\Gamma = \sum_p |\gamma_p| \tag{3.6}$$

On behalf of (2.8) Γ cannot increase with time when none of the w_{pq} are negative. Thus the final bound becomes independent of j:

$$|\delta_j| \leq \delta \tag{3.7}$$

$$\delta = |\hat{\phi}(k\ell)| \Gamma \left(\frac{\sqrt{2}}{M!} {d_0}^{M-2} \sqrt{2\nu\Delta t}^M k^M + \frac{1}{2^N N!} \sqrt{2\nu\Delta t}^{2N} k^{2N} \right) \tag{3.8}$$

Starting from an initial set of vortex blobs $\hat{\omega}_0$, the solution after $J = t/\Delta t$ steps follows as

$$\hat{\omega}_J = \hat{\omega}_0 e^{-k^2\nu t} + \delta_0 e^{-k^2\nu(J-1)\Delta t} + \delta_1 e^{-k^2\nu(J-2)\Delta t} + \ldots + \delta_{J-2} e^{-k^2\nu\Delta t} + \delta_{J-1} \tag{3.9}$$

Since the first term of the right hand side describes the exact solution, the error can be bounded by

$$|\hat{\omega}_J - \hat{\omega}_0 e^{-k^2\nu t}| \leq \delta \left(e^{-k^2\nu(J-1)\Delta t} + e^{-k^2\nu(J-2)\Delta t} + \ldots + e^{-k^2\nu\Delta t} + 1 \right) \tag{3.10}$$

or after approximating the geometric series,

$$|\hat{\omega}_J - \hat{\omega}_0 e^{-k^2\nu t}| \leq \frac{\delta}{1 - e^{-k^2\nu\Delta t}} \tag{3.11}$$

or

$$|\hat{\omega}_J - \hat{\omega}_0 e^{-k^2\nu t}| \leq \frac{2\delta}{k^2\nu\Delta t} \tag{3.12}$$

provided

$$\Delta t \leq \frac{1}{2k^2\nu} \tag{3.13}$$

Substitution of (3.8) in (3.12) yields the final error bound

$$|\hat{\omega}_J - \hat{\omega}_0 e^{-k^2\nu t}| \leq |\hat{\phi}(k\ell)|\Gamma \left(\frac{4\sqrt{2}}{M!} {d_0}^{M-2} \sqrt{2\nu\Delta t}^{M-2} k^{M-2} + \frac{4}{2^N N!} \sqrt{2\nu\Delta t}^{2N-2} k^{2N-2} \right) \tag{3.14}$$

To show that the error (3.15) can be reduced to any desired amount, we first introduce a value K of k so that the region $k \geq K$ contributes less than half the desired error to the L_2-norm of (3.14). The exponential decay of the function $\hat{\phi}(k\ell)$ allows such a value for K to be found. Further, in the region $k \leq K$, the error (3.14) decreases uniformly with Δt so that its L_2 norm also can be made less than half the desired error.

To simply (3.14), we can introduce a mesh length scale

$$\ell_m = d_0\sqrt{2\nu\Delta t} = \max(|\mathbf{r}_q - \mathbf{r}_p|) \tag{3.15}$$

and a diffusion length scale

$$\ell_d = \sqrt{2\nu\Delta t} \tag{3.16}$$

in which case (3.14) reads

$$|\hat{\omega}_J - \hat{\omega}_0 e^{-k^2\nu t}| \leq |\hat{\phi}(k\ell)|\Gamma \left(\frac{4\sqrt{2}}{M!} (\ell_m k)^{M-2} + \frac{4}{2^N N!} (\ell_d k)^{2N-2} \right) \tag{3.17}$$

Each mode is therefor integrated accurately when both the mesh and diffusion length scale are small compared to the wave length $1/k$.

However, small errors in the modes do not necessarily imply a small total square error. To obtain an integrated square error of order $\epsilon^2\Gamma^2$, we would need

$$K = \frac{C_K}{\ell} \tag{3.18}$$

on account of (2.3), where C_K is only logarithmically large in ϵ. Then the mesh and diffusion length scales must be of order

$$\ell_m \approx C_m \epsilon^{\frac{1}{M-2}} \ell^{\frac{M-1}{M-2}} \tag{3.19}$$

$$\ell_d \approx C_d \epsilon^{\frac{1}{2N-2}} \ell^{\frac{2N-1}{2N-2}} \tag{3.20}$$

4. Numerical test for the one-dimensional heat equation

To test the performance of the proposed algorithm, an appropriate one-dimensional exact solution is the diffusing vortex,

$$\omega = \frac{1}{\sqrt{t}} e^{-x^2/4\nu t} \tag{4.1}$$

The corresponding circulation, given as the integral with respect to x, is

$$G = \sqrt{\nu\pi}\,\mathrm{erfc}\left(-\frac{x}{\sqrt{4\nu t}}\right) \tag{4.2}$$

To simulate the solution using the random walk method, initially 64 vortices of equal strength were placed at the origin. At each time step the vortices are given a random displacement proportional to $\sqrt{2\nu\Delta t}$ selected from a data base of 8000 gaussian random numbers. The deviations from the exact solution are appreciable for longer times. Figure 1b shows the values of the first vorticity moment

$$C_1 = \sum_p \gamma_p x_p \tag{4.3}$$

which should be zero, reflecting conservation of the position of the center of vorticity. However, the lower curve in figure 1b shows that a significant drift of the center does occur. Further the ratio

$$C_2 = \frac{\sum_p \gamma_p x_p^2}{4\nu t\sqrt{\nu\pi}} \tag{4.4}$$

also shows significant deviations from its exact, unit, value.

The drift of the center of vorticity may be reduced by slightly adjusting the distribution of the 8000 random numbers to give exactly zero mean. Under that condition, the results appear as shown in figure 2. Yet, there is no real justification for this correction.

In the redistribution procedure, initially a single vortex was placed at the origin. At each time step, vortices were added to satisfy the non-negativity condition. The procedure followed here was to take the neighborhood of a vortex as all vortices within a distance $2\ell_d$. A first attempt was made to satisfy (2.8) through (2.11) in this neighborhood with a non-negative solution. If this failed, next an attempt was made to satisfy (2.8) through (2.10) only. If again no non-negative solution could be found, vortices were added until a non-negative solution did result.

In adding vortices, a simple and effective procedure is to add the new vortex at a distance $\sqrt{3}l_d$ from the considered vortex. The side at which to add the vortex is chosen by maximizing the distance between the new vortex and the closest existing vortex in the neighborhood. The results are shown in figure 3, and are clearly much more accurate than the random walk procedure. It can be shown that a non-negative solution to equations (2.8) through (2.10) must exist when there are two vortices in the neighborhood at opposite sides of the considered vortex, at a geometric mean distance of at least ℓ_d. Thus the number of vortices that needs to be added in the described procedure is at most 2.

In fact the procedure of figure 3 simply generates a uniform mesh, and the redistribution algorithm then generates second order accurate explicit finite difference formulae, explaining the good results.

Such an effective procedure would be much more difficult to develop in two or more dimensions, and would not hold in the presence of convection. Therefor, to get a more realistic assessment of the redistribution algorithm, we need to destroy the mesh regularity. This was done by displacing each added vortex a random distance away from the nominal $\sqrt{3}\ell_d$ value.

Using a gaussian random displacement with a standard deviation of $0.83\ell_d$, (and ignoring values outside the range from 0 to $2\ell_d$), the results of figure 4 were obtained. In this case the high frequency errors predicted by the analysis of the previous section are observed. It is noted that figure 4 has not been smoothed; the size of the vortex blobs is zero. In fact, the integration of the vorticity (4.2) ensures that L_2-convergence occurs regardless of smoothing.

To make the position at which the vortices are added essentially random, the random displacement was increased to $\sqrt{3}\ell_d$. Results for various time steps are shown in figures 5a through c. The results are very satisfactory.

A second example, in which the random walk method experiences considerably more difficulty, is the diffusing dipole,

$$\omega = \frac{-x}{2\nu t\sqrt{t}} e^{-x^2/4\nu t} \tag{4.5}$$

Figure 6a compares the exact solution with the random walk results, while figure 6b presents the invariants

$$D_1 = -\frac{\sum_p \gamma_p x_p}{2\sqrt{\nu\pi}} \tag{4.6}$$

$$D_2 = \sum_p \gamma_p x_p^2 \tag{4.7}$$

Figure 7 shows that this case is again integrated much more accurately with the redistribution method, even using randomly chosen points.

1. L. L. Van Dommelen and E. A. Rundensteiner, "Fast solution of the two-dimensional Poisson equation with point-wise forcing", *Journal of Computational Physics*, to appear.

2. L. L. Van Dommelen "Least-maximum solution of underdetermined linear systems", in preparation.

Figures

1. Random walk results for a diffusing one-dimensional vortex. (a) Circulation. (b) Invariants.

2. As figure 1, but with corrected random numbers.

3. Performance of the redistribution algorithm for the one-dimensional diffusing vortex when well chosen vortex positions are used.

4. Similar to figure 3, but with less well chosen vortex positions.

5. Similar to figures 3 and 4, but with randomly chosen vortex positions. (a) $\Delta t = 0.125$ (b) $\Delta t = 0.0625$ (c) $\Delta t = 0.03125$

6. Random walk results for a diffusing dipole. (a) Circulation. (b) Invariants.

7. Circulation for the diffusing dipole using the redistribution method.