

Spectrum Sharing Between Wireless Networks

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Abstract—We consider the problem of two wireless networks operating on the same (presumably unlicensed) frequency band. Pairs within a given network cooperate to schedule transmissions, but between networks there is competition for spectrum. To make the problem tractable, we assume transmissions are scheduled according to a random access protocol where each network chooses an access probability for its users. A game between the two networks is defined. We characterize the Nash Equilibrium behavior of the system. Three regimes are identified: one in which both networks simultaneously schedule all transmissions, one in which the denser network schedules all transmissions and the sparser only schedules a fraction, and one in which both networks schedule only a fraction of their transmissions. The regime of operation depends on the path loss exponent α , the latter regime being desirable but attainable only for $\alpha > 4$. This suggests that in certain environments, rival wireless networks may end up naturally cooperating. To substantiate our analytical results, we simulate a system where networks iteratively optimize their access probabilities in a greedy manner. We also discuss a distributed scheduling protocol that employs carrier sensing and demonstrate via simulations that again a near cooperative equilibrium exists for sufficiently large α .

Index Terms—Carrier sensing, game theory, Nash equilibrium (N.E.), price of anarchy, random access, spectrum sharing, wireless ad hoc networks.

I. INTRODUCTION

THE recent proliferation of networks operating on unlicensed bands, most notably 802.11 and Bluetooth, has stimulated research into the study of how different systems competing for the same spectrum interact [1]. Communication on unlicensed spectrum is desirable essentially because it is free, but users are subject to random interference generated by the transmissions of other users. Most research to date has assumed devices have no natural incentive to cooperate with one another. For instance, a wireless router in one apartment is not concerned about the interference it generates in a neighboring apartment. Following from this assumption, various game-theoretic formulations have been used to model the interplay between neighboring systems [2]–[6]. An important conclusion stemming from this body of work is that for single-stage games, the Nash equilibria (N.E.) are typically unfavorable, resulting in

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inefficient allocations of resources to users. A quintessential example is the following. Consider a system where a pair of competing links is subjected to white noise and all cross gains are frequency-flat. Suppose the transmitters wish to select a one-time power allocation across frequency subject to a constraint on the total power expended (this problem is studied in [7], [16], and [17], where it is referred to as the *Gaussian Interference Game*). It is straightforward to reason (via a waterfilling argument) that the selection by both users of frequency-flat power allocations, each occupying the entire band, constitutes a N.E. This *full-spread* power allocation can be extremely inefficient. Consider a symmetric system where the cross gains and direct gains are equal. At high SNR, each link achieves a throughput of only 1 b/s/Hz, instead of $\frac{1}{2} \log_2(1 + \text{SNR})$ b/s/Hz, which would be obtained if the links cooperated by occupying orthogonal halves of the spectrum. At a SNR of 30 dB, the throughput ratio between cooperative behavior and this full-spread N.E. behavior, referred to as the *price of anarchy*, is about 5. This example highlights an important point in relation to single-stage games between competing wireless links: Users typically have an incentive to occupy all of the available resources.

In this work a different approach is taken. Rather than assuming total anarchy—that is, competition between *all* wireless links—we instead assume competition only between wireless links belonging to *different networks*. Wireless links belonging to the same network are assumed to *cooperate*. In short, we assume competition on the network level, not on the link level. In a practical setting this may represent the fact that neighboring wireless systems are produced by the same manufacturer or are administered by the same network operator. Alternatively, one may view the competition as being between coalitions of users [8].

To make the problem analytically tractable but still retain its underlying mechanics, we assume each network operates under a random access protocol, where users from a given network access the channel independently but with the same probability. Analysis of random access protocols provides intuition for the behavior of systems operating under more complex protocols, as the access probability can broadly be interpreted as the average degrees of freedom each user occupies. For the case of competition on the link level, game-theoretic research of random access protocols such as ALOHA have been conducted in [9] and [10]. In our model, each network has a different density of nodes and chooses its access probability to maximize average throughput per user. Note this access protocol is essentially identical to one in which users select a random fraction of the spectrum on which to communicate. Thus, an access probability of one corresponds to a full-spread power allocation.

We first assume all links in the system have the same transmission range and afterward show that the results are only trivially modified if each link is assumed to have an i.i.d. random

transmission range. We characterize the N.E. of this system for a fixed-rate model, where all users transmit at the same data rate. We show that unlike the case of competing links, a N.E. always exists and is *unique*. Furthermore, for a large range of typical parameter values, the N.E. is not full-spread—nodes in at least one network occupy only a fraction of the bandwidth. We also identify two modes, delineated by the path loss exponent α . For $\alpha > 4$, the N.E. behavior is distinctly different than for $\alpha < 4$ and possesses pseudocooperative properties. Following this, we show that the picture for the variable-rate model, in which users individually tailor their transmission rates to match the instantaneous channel capacity, remains unchanged. Before concluding, we present simulation results for the behavior of the system when the networks employ a greedy algorithm to optimize their throughput, operating under both a random access protocol and a carrier sensing protocol.

In Section II, we formulate the system model explicitly. In Section III-A, we introduce the random access protocol and analyze its N.E. behavior in the fixed-rate model. In Section III-B, we analyze the variable rate model. In Section IV, we extend our results to cover the case of variable transmission ranges. Section V presents simulation results and the carrier sensing protocol. Section VI summarizes and suggests extensions. The Appendix contains proofs of the main theorems presented.

II. PROBLEM SETUP

Consider two wireless networks consisting of $n\lambda_1$ and $n\lambda_2$ tx–rx pairs, respectively. Without loss of generality, we will assume $\lambda_1 \leq \lambda_2$. The transmitting nodes are uniformly distributed at random in an area of size n . To avoid boundary effects, suppose this area is the surface of a sphere. Thus, λ_i is the density of transmitters (or receivers) in network i . For each transmitter, the corresponding receiver is initially assumed to be located at a fixed range of d meters with uniform random bearing. Time is slotted, and all users are assumed to be time-synchronized.

Both networks operate on the same band of (presumably unlicensed) spectrum, and at each time slot, a subset of tx–rx pairs are simultaneously scheduled in each network. When scheduled, a tx–rx pair uses all of the spectrum. It is generally desirable to schedule neighboring tx–rx pairs in different time slots. This scheduling model is a form of time division multiplexing (TDM), but is more or less analogous to a frequency division multiplexing (FDM) model where each tx–rx pair is allocated a subset of the spectrum (typically overlapping in some way with other tx–rx pairs in the network).

Transmitting nodes are full buffer in that they always have data to send. Transmissions are assumed to use Gaussian codebooks, and interference from other nodes is treated as noise. Initially, we analyze the model where all transmissions in network i occur at a common rate of $\log(1 + \beta_i)$. We refer to β as the *target SINR*. Thus, a transmission in network i is successful iff $\text{SINR} > \beta_i$. Later, we explore the model where transmission rates are individually tailored to match the instantaneous capacities of the channels. The signal power attenuates according to a power law with *path loss exponent* $\alpha > 2$. We assume a high-SNR or *interference-limited* scenario, where the thermal noise is insignificant relative to the received power of interfering nodes, and thus refer to the SIR as the SIR. For a given

realization of the node locations, the time-averaged throughput achieved by the j th tx–rx pair in network i is then

$$\bar{R}_j = f_j \mathbb{P}(\text{SIR}_j(t) > \beta_i) \log(1 + \beta_i)$$

per complex d.o.f., where f_j is the fraction of time the j th tx–rx pair is scheduled. The notation $\mathbb{P}(X)$ denotes the probability of the event X occurring. The average (represented by the bar above the R) is essentially taken over the distribution of the interference as different subsets of transmitters are scheduled at different times.

As for $\alpha > 2$, the bulk of the interference is generated by the strongest interferer. To make the problem tractable, we compute the SIR as the receive power of the desired signal divided by the receive power of the *nearest* interferers signal. We refer to this as the *Dominant Interferer* assumption. This assumption is also employed in the analysis of similar networks in [11]–[14]. Further justification is provided by way of simulations presented in Section V. These illustrate that our results remain unchanged when the assumption is removed.

Denote the range of the nearest interferer to the j th receiver at time t by $r_j(t)$. Then

$$\text{SIR}_j(t) = \frac{d^{-\alpha}}{r_j^{-\alpha}(t)}.$$

The metric of interest to each network is its expected time-averaged rate per user

$$\mathbb{E}\bar{R} = \mathbb{E}_g [f_j \mathbb{P}(\text{SIR}_j(t) > \beta_i) \log(1 + \beta_i)].$$

The subscript g indicates this expectation is taken over the geographic distribution of the nodes. As the setup is statistically symmetric, this metric is equivalent to the expected *sum rate* of the system, divided by $n\lambda_i$, in the limit $n \rightarrow \infty$.

III. RANDOM ACCESS PROTOCOL

A. Fixed-Rate Model

Suppose each network uses the following random access protocol. At each time slot, each link is scheduled i.i.d. with probability p_i . The packet size is $\log(1 + \beta_i)$ for all communications in network i . The variables β_i are optimized over.

Let us first compute the optimal access probability for the case of a single network operating in isolation on a licensed band as a function of the node density and the transmission range. This problem has recently been studied independently in [11]–[14] with equivalent results derived. In [15], similar results are derived for the case where the SIR is computed based on all interferers, not just the nearest.

Let the r.v. $N_j(x)$ denote the number of interfering transmitters within range x of the j th receiver.

$$\begin{aligned} \mathbb{E}\bar{R} &= \mathbb{E}_g \left[f_j \mathbb{P}(r_j(t) > \beta_i^{1/\alpha} d) \log(1 + \beta_i) \right] \\ &= p_i \sum_{k=1}^{n\lambda_i - 1} \mathbb{P}(N_j(\beta_i^{1/\alpha} d) = k) (1 - p_i)^k \log(1 + \beta_i) \end{aligned}$$

$$\begin{aligned}
 &= p_i \sum_{k=1}^{n\lambda_i-1} \left(\frac{\pi\beta_i^{2/\alpha} d^2}{n} \right)^k \left(1 - \frac{\pi\beta_i^{2/\alpha} d^2}{n} \right)^{n\lambda_i-k-1} \\
 &\quad \times \binom{n\lambda_i-1}{k} (1-p_i)^k \log(1+\beta_i) \\
 &= p_i \left(1 - p_i \frac{\pi\beta_i^{2/\alpha} d^2}{n} \right)^{n\lambda_i-1} \log(1+\beta_i) \\
 &\rightarrow p_i \log(1+\beta_i) e^{-\pi\lambda_i d^2 p_i \beta_i^{2/\alpha}}
 \end{aligned}$$

in the limit $n \rightarrow \infty$.

In order to obtain better insight into the problem at hand, a change of variables is required. We refer to the set of all points within the transmission range as the *transmission disc*. The quantity $\pi\lambda_i d^2$ is the *average number of nodes (tx or rx) per transmission disc*. We often refer to it simply as the *number of nodes per disc* and represent it by the symbol

$$N_i \triangleq \pi\lambda_i d^2.$$

Assume N_i is larger than a certain threshold (we make this precise later). Maximizing over the access probability yields

$$\overline{R} \rightarrow \frac{\log(1+\beta_i)}{N_i \beta_i^{2/\alpha}} e^{-1} \quad (1)$$

with the optimal access probability being

$$p_i^* = \frac{1}{N_i \beta_i^{2/\alpha}}. \quad (2)$$

One can further optimize over the target SIR so that β_i is replaced by β_i^* in the above two equations. Inspection of (1) reveals that the optimal target SIR is a function of α alone. Thus, if we define the quantity

$$\Lambda_i \triangleq N_i p_i$$

Λ_i^* will be a constant independent of N_i . The quantity Λ_i represents the *average number of (simultaneous) transmissions per transmission disc*. We sometimes refer to it simply as the *transmit density*. Whereas the domain of p_i is $[0, 1]$, the domain of Λ_i is $[0, N_i]$. Thus, we see that for N_i sufficiently large, the access probability should be set such that an optimal number of transmissions per disc is achieved. What is this optimal number? What is the optimal target SIR?

For the purposes of optimizing (1), define the function $\Lambda^*(\alpha)$ as the unique solution of the following equation:

$$\frac{\alpha}{2} = \left(1 + \Lambda^{*\alpha/2} \right) \log \left(1 + \frac{1}{\Lambda^{*\alpha/2}} \right). \quad (3)$$

This equation comes from the observation that maximizing (1) over β_i is equivalent to maximizing $\Lambda \log(1 + \Lambda^{-\alpha/2})$ over Λ . A plot of $\Lambda^*(\alpha)$ is given in Fig. 1. So as to avoid confusion, note that the symbol Λ^* represents a predefined function, not necessarily the same as the symbol Λ_i^* , which is a variable. As

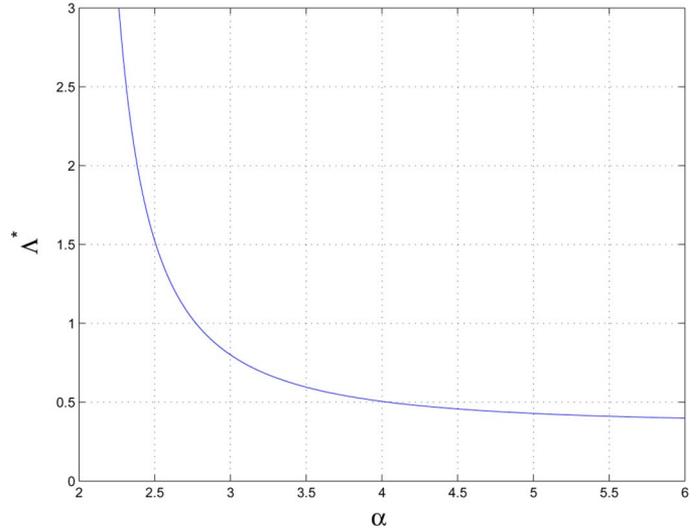


Fig. 1. Optimal average number of transmissions per transmission disc as a function of the path loss exponent.

(1) is smooth and continuous with a unique maxima, by setting its derivative to zero, we find that the optimal target SIR is $\beta_i^* = \Lambda^{*-\alpha/2}$, and the optimal number of transmissions per disc is $\Lambda_i^* = \Lambda^*$, when N_i is larger than a certain threshold.

When N_i is smaller than this threshold, there are not enough tx-rx pairs to reach the optimal number of transmissions per disc, even when all of them are simultaneously scheduled. In this case, the solution lies on the boundary with $p_i^* = 1$. This corresponds to the scenario where the transmission range is short relative to the node density such that tx-rx pairs function as if in isolation. It is intuitive that in this case all transmissions in the network will be simultaneously scheduled. Our discussion is summarized in the following theorem.

Theorem III-1: (Optimal Access Probability) For a single network operating in isolation under the random access protocol, when $N_i > \Lambda^*$, the optimal access probability is $p_i^* = \frac{\Lambda^*}{N_i}$, where Λ^* is given by the unique solution of (3). The optimal target SIR is $\beta_i^* = \Lambda^{*-\alpha/2}$.

When $N_i \leq \Lambda^*$, $p_i^* = 1$ and β_i^* is given by the unique solution to

$$\frac{\alpha}{2N_i \beta_i^{2/\alpha}} = \left(1 + \frac{1}{\beta_i^*} \right) \log(1 + \beta_i^*).$$

The expression for β_i^* at the end of this theorem is the first-order stationarity condition for maximizing the last equation over β_i with $p_i = 1$. The region satisfying $N_i > \Lambda^*$ is referred to as the *partial reuse regime*. The complement region is referred to as the *full reuse regime*. Note the optimal access probability of the above theorem is equivalent to the results of [11, Section IV.B], and those discussed under the title “Maximum Achievable Spatial Throughput and TC” in [13, p. 4137].

Now, we perform the same computation for the case where both networks operate on the same unlicensed band. In this case, there is both *intranetwork* and *internetwork* interference. It is

straightforward to extend the above analysis to show that for network i

$$\mathbb{E}\bar{R}_i \rightarrow \frac{\Lambda_i}{N_i} \log(1 + \beta_i) e^{-(\Lambda_1 + \Lambda_2)\beta_i^{2/\alpha}}$$

in the limit $n \rightarrow \infty$. For a given Λ_2 , the first network can optimize Λ_1 and β_1 , and vice-versa. That is each network can iteratively adjust its access probability and target SIR in response to the other network. In this sense, a game can be defined between the two networks. A strategy for network i is a choice of $\Lambda_i \in [0, N_i]$ and $\beta_i > 0$. Its payoff function (also referred to as *utility function*) is the limiting form of $\mathbb{E}\bar{R}_i$ times N_i

$$U_1((\Lambda_1, \beta_1), (\Lambda_2, \beta_2)) = \Lambda_1 \log(1 + \beta_1) e^{-(\Lambda_1 + \Lambda_2)\beta_1^{2/\alpha}} \quad (4)$$

$$U_2((\Lambda_1, \beta_1), (\Lambda_2, \beta_2)) = \Lambda_2 \log(1 + \beta_2) e^{-(\Lambda_1 + \Lambda_2)\beta_2^{2/\alpha}}. \quad (5)$$

Here, we have scaled the throughput by N_i to emphasize the simple form of the payoff functions. At first glance, this setup seems desirable, but there is a redundancy in the way the strategy space has been defined. The problem is that the variable β_i only appears in U_i and thus should be optimized over separately rather than being included as part of the strategy. This leads to the following game setup.

Definition III-2: (Random Access Game) A strategy for network i in the Random Access Game is a choice of $\Lambda_i \in [0, N_i]$. The payoff functions are

$$U_1(\Lambda_1, \Lambda_2) = \max_{\beta_1 > 0} \Lambda_1 \log(1 + \beta_1) e^{-(\Lambda_1 + \Lambda_2)\beta_1^{2/\alpha}}$$

$$U_2(\Lambda_1, \Lambda_2) = \max_{\beta_2 > 0} \Lambda_2 \log(1 + \beta_2) e^{-(\Lambda_1 + \Lambda_2)\beta_2^{2/\alpha}}.$$

The above formulation is intuitively appealing as a network's choice of access probability constitutes its entire strategy. If we could explicitly solve the maximization problems, the variables β_i would be removed altogether. When Λ_1 and Λ_2 are large, this can be done, and

$$U_1(\Lambda_1, \Lambda_2) \approx \frac{\Lambda_1}{(\Lambda_1 + \Lambda_2)^{\alpha/2}} \cdot (\alpha/2)^{\alpha/2} e^{-\alpha/2} \quad (6)$$

$$U_2(\Lambda_1, \Lambda_2) \approx \frac{\Lambda_2}{(\Lambda_1 + \Lambda_2)^{\alpha/2}} \cdot (\alpha/2)^{\alpha/2} e^{-\alpha/2} \quad (7)$$

but, in general, it is not possible. Instead, since we are only interested in analyzing the Nash equilibrium (N.E.) or equilibria of this game, we do the following.

Observe that the objective function within the maximization is smooth and continuous. This enables the order of maximization to be swapped. That is, for a given Λ_2 , we first maximize over Λ_1 in (4) and then over the β_1 ; likewise for (5). The benefit of this approach is that the maximizing Λ_i can be explicitly expressed as a function of β_i . This was demonstrated earlier for the single-network scenario. The resulting expressions are

$$U_1(\Lambda_1, \Lambda_2) = \begin{cases} \Lambda_1 \log\left(1 + \Lambda_1^{-\alpha/2}\right) e^{-\Lambda_2/\Lambda_1 - 1}, & \Lambda_1 < N_1 \\ \max_{\beta_1 > 0} \log(1 + \beta_1) e^{-(N_1 + \Lambda_2)\beta_1^{2/\alpha}}, & \Lambda_1 = N_1 \end{cases}$$

$$U_2(\Lambda_1, \Lambda_2) = \begin{cases} \Lambda_2 \log\left(1 + \Lambda_2^{-\alpha/2}\right) e^{-\Lambda_1/\Lambda_2 - 1}, & \Lambda_2 < N_2 \\ \max_{\beta_2 > 0} \log(1 + \beta_2) e^{-(N_2 + \Lambda_1)\beta_2^{2/\alpha}}, & \Lambda_2 = N_2. \end{cases}$$

The set of N.E. of the above game and their corresponding values of U_1 and U_2 are identical to those of the Random Access Game. Inspection of the above equations reveals a further simplification of the problem is at hand: The set of N.E. of the above game are identical to the set of N.E. of the following game (though the values of U_1 and U_2 at the equilibria may be different).

Definition III-3: (Transformed Random Access Game) A strategy for network i in the Transformed Random Access Game is a choice of $\Lambda_i \in [0, N_i]$. The payoff functions are

$$U_1(\Lambda_1, \Lambda_2) = \Lambda_1 \log\left(1 + \Lambda_1^{-\alpha/2}\right) e^{-\frac{\Lambda_2}{\Lambda_1} - 1}$$

$$U_2(\Lambda_1, \Lambda_2) = \Lambda_2 \log\left(1 + \Lambda_2^{-\alpha/2}\right) e^{-\frac{\Lambda_1}{\Lambda_2} - 1}.$$

We now analyze the N.E. of the Random Access Game by analyzing the N.E. of the Transformed Random Access Game. The first question of interest is whether or not there exists a N.E. It turns out a unique N.E. always exists, but its nature depends crucially on the path loss exponent. There are two *modes*, $2 < \alpha < 4$ and $\alpha > 4$. We start with the first. Recall we have assumed $\lambda_1 \leq \lambda_2$.

Theorem III-4: (Random Access N.E. for $2 < \alpha < 4$) For $2 < \alpha < 4$, the unique N.E. occurs at $\Lambda_1^* = N_1$ and Λ_2^* defined by either the solution of

$$N_1 = \Lambda_2 \left(\frac{\alpha}{2 \left(1 + \Lambda_2^{\alpha/2}\right) \log\left(1 + \Lambda_2^{-\alpha/2}\right)} - 1 \right) \quad (8)$$

or N_1 , whichever is smaller.

The N.E. described in Theorem III-4 occurs on the boundary of the strategy space. This is because for $2 < \alpha < 4$, each network tries to set its number of transmissions per disc higher than the other (see the proof of the theorem). The equilibrium is then only attained when at least one network has maxed out and scheduled all of its transmissions simultaneously.

The N.E. can be better understood when $N_1 \gg 1$, corresponding to the case in which transmissions span several intermediate nodes.

Theorem III-5: (Random Access N.E. for $2 < \alpha < 4$ and $N_1 \gg 1$) In the limit $N_1 \rightarrow \infty$, the N.E. occurs at

$$(\Lambda_1^*, \Lambda_2^*) = \begin{cases} (N_1, N_2), & N_1 \leq N_2 \leq \frac{2}{\alpha-2} N_1 \\ \left(N_1, \frac{2}{\alpha-2} N_1\right), & \frac{2}{\alpha-2} N_1 \leq N_2. \end{cases}$$

This result stems from using the limiting form $\log(1+x) \rightarrow x$ as $x \rightarrow 0$ in the utility functions U_1 and U_2 , as was done in (6) and (7). From it, we see that if the denser network has more than $\approx 2/(\alpha-2)$ times as many nodes as its rival, the N.E. will correspond to partial reuse, i.e., the denser network will only occupy a fraction of the total available bandwidth. This is in stark contrast to the case of competing *individual transmissions* where

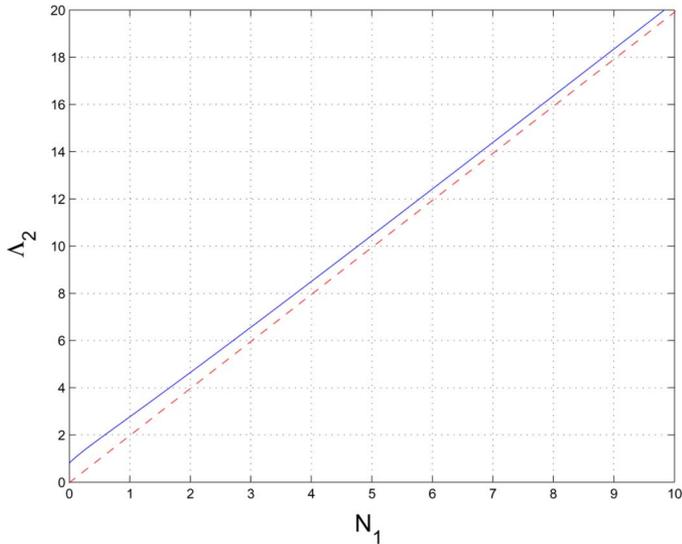


Fig. 2. The solid line represents the solution to (8) for $\alpha = 3$. The N.E. value Λ_2^* is equal to the minimum of this line and N_2 . The limiting solution used in Theorem III-5 is plotted as a dashed line.

the N.E. typically corresponds to a full spread, i.e., both competing links spread their power evenly across the entire bandwidth. The limit result of Theorem III-5 is plotted in Fig. 2 as a dashed line.

We now investigate the average throughput at equilibrium for the mode $2 < \alpha < 4$. We define the metric

$$U_e = U_1(\Lambda_1^*, \Lambda_2^*) + U_2(\Lambda_1^*, \Lambda_2^*).$$

This quantity has a natural interpretation. Recall $\mathbb{E}\bar{R}_i$ is the average throughput per tx-rx pair, and N_i is the average number of tx-rx pairs per transmission disc in network i . Thus, $U_i = N_i \mathbb{E}\bar{R}_i$ is the *average throughput per transmission disc* in network i . This is the average number of bits successfully received in network i within an area of size πd^2 per time slot, per d.o.f. The quantity U_e is then the average throughput per transmission disc in the system, that is, the average number of bits successfully received *in both networks* within an area of size πd^2 per time slot, per d.o.f.

Theorem III-6: In the limit $N_1 \rightarrow \infty$

$$U_e \rightarrow \begin{cases} \frac{c_1(\alpha)}{N_1^{\alpha/2-1}}, & N_1 \leq N_2 \leq \frac{2}{\alpha-2}N_1 \\ \frac{c_2(\alpha)}{(N_1+N_2)^{\alpha/2-1}}, & \frac{2}{\alpha-2}N_1 \leq N_2. \end{cases}$$

where $c_1(\alpha) = (\alpha/2 - 1)^{\alpha/2-1} (\alpha/2)e^{-\alpha/2}$ and $c_2(\alpha) = (\alpha/2)^{\alpha/2} e^{-\alpha/2}$.

The important property of this result is that as the number of nodes per transmission disc increases, U_e decreases roughly like $1/(N_1 + N_2)^{\alpha/2-1}$. Let us compare this to the average throughput per transmission disc in the cooperative case, that is when the two networks behave as if they were a single network with $N_1 + N_2$ nodes per disc. From (1), this average throughput per disc is

$$U_c = \Lambda^* \log(1 + \Lambda^{*\alpha/2})$$

which is independent of the number of nodes per disc. Thus, as the number of nodes per disc grows, so does the price of anarchy

$$\frac{U_c}{U_e} = O\left(N_1^{\alpha/2-1}\right).$$

For $\alpha > 4$, the N.E. behavior is different. Whereas the solution always lies on the boundary for $2 < \alpha < 4$, for $\alpha > 4$ it typically does not.

Theorem III-7: (Random Access N.E. for $\alpha > 4$) For $\alpha > 4$, the unique N.E. occurs at

$$(\Lambda_1^*, \Lambda_2^*) = \left(\sqrt{\Lambda^*(\alpha/2)}, \sqrt{\Lambda^*(\alpha/2)}\right)$$

if $\sqrt{\Lambda^*(\alpha/2)} < N_1$; otherwise $\Lambda_1^* = N_1$, and Λ_2^* is defined by either the solution of (8) or N_2 , whichever is smaller.

A plot of $\sqrt{\Lambda^*(\alpha/2)}$ versus α is given in Fig. 6. The condition $\sqrt{\Lambda^*(\alpha/2)} < N_1$ corresponds to network 1 having more than $\sqrt{\Lambda^*(\alpha/2)}$ nodes per transmission disc. We refer to this as the *partial/partial reuse regime*.

The interpretation of Theorem III-7 is that for $\alpha > 4$ in the partial/partial reuse regime, the solution lies in the strict interior of the strategy space. This is because on the boundary of the space network i can improve its throughput by undercutting the transmit density of network j , i.e., setting $\Lambda_i < \Lambda_j$. The symmetry of the N.E. ($\Lambda_1^* = \Lambda_2^*$) then follows by observing that the utility functions are symmetric and the solution is unique.

There is a cooperative flavor to this equilibrium in that both networks set their transmission densities to the same level, and this level is comparable to the optimal single-network density $\Lambda^*(\alpha)$. Moreover, the equilibrium level does not grow with the number of nodes per transmission disc as it does for $2 < \alpha < 4$. Actual cooperation between networks corresponds to setting the access probability based on (2), taking into account that the effective node density is $\lambda_1 + \lambda_2$. Thus, the cooperative solution is

$$(\Lambda_1^*, \Lambda_2^*) = \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} \Lambda^*, \frac{\lambda_2}{\lambda_1 + \lambda_2} \Lambda^*\right).$$

Under cooperation, the average throughput per transmission disc at the N.E. is [from (1)]

$$U_c = \frac{1}{e} \Lambda^*(\alpha) \log\left(1 + \frac{1}{\Lambda^*(\alpha)^{\alpha/2}}\right).$$

Under cooperation in the partial/partial reuse regime, it is

$$U_e = \frac{2}{e^2} \sqrt{\Lambda^*(\alpha/2)} \log\left(1 + \frac{1}{\Lambda^*(\alpha/2)^{\alpha/4}}\right).$$

The price of anarchy is the ratio of these two quantities (U_e/U_c). Comparing the two modes, we see that while for $2 < \alpha < 4$ the price of anarchy grows in an unbounded fashion with the number of nodes per transmission disc, for $\alpha > 4$ the price of anarchy in the partial/partial reuse regime is a constant depending only on α . This constant is plotted versus α in Fig. 4.

We now summarize the equilibria results. There are three regimes.

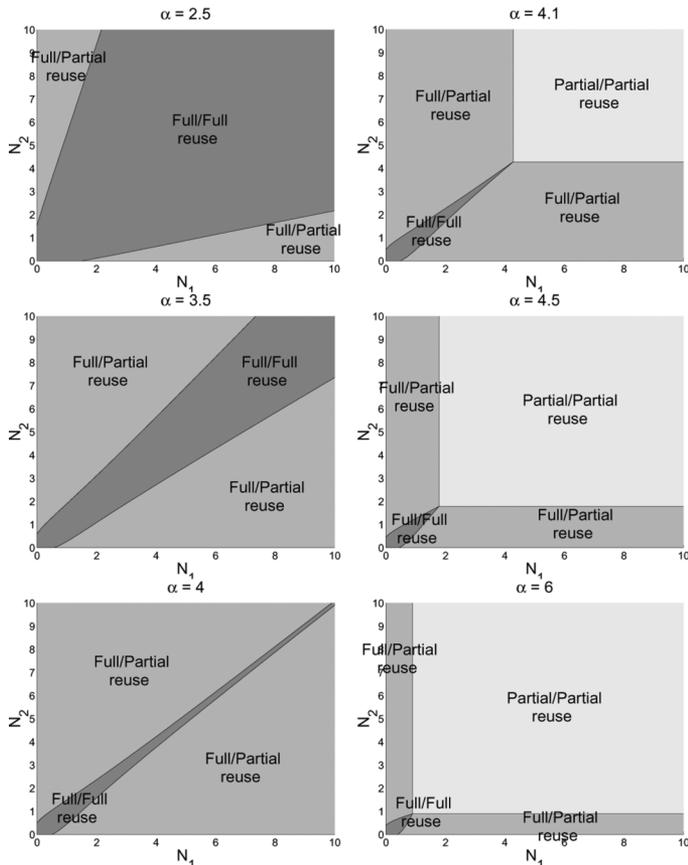


Fig. 3. These plots can be used to determine which regime the N.E. is in. The x-axis and y-axis correspond to the average number of nodes per transmission disc in networks 1 and 2, respectively. Note the lower left vertex of the partial/partial reuse regime always occurs at $(\sqrt{\Lambda^*(\alpha/2)}, \sqrt{\Lambda^*(\alpha)})$ and the intersection of the full/full reuse regime with the axes always occurs at $(\Lambda^*(\alpha), 0)$ and $(0, \Lambda^*(\alpha))$.

1) *Full/Full reuse*

$$N_1 \leq \sqrt{\Lambda^*(\alpha/2)} \quad \text{and} \quad N_2 \leq \frac{\alpha/2 \left(1 + N_2^{\alpha/2}\right) \log\left(1 + N_2^{-\alpha/2}\right) - 1}{\sqrt{\Lambda^*(\alpha/2)}}.$$

— Both networks schedule all transmissions.

2) *Full/Partial reuse*

$$N_1 \leq \sqrt{\Lambda^*(\alpha/2)} \quad \text{and} \quad N_1 > \frac{\alpha/2 \left(1 + N_2^{\alpha/2}\right) \log\left(1 + N_2^{-\alpha/2}\right) - 1}{\sqrt{\Lambda^*(\alpha/2)}}.$$

— Denser network schedules all transmissions; sparser schedules only a fraction.

3) *Partial/Partial reuse*

$$N_1 > \sqrt{\Lambda^*(\alpha/2)}.$$

— Both networks schedule only a fraction of their transmissions.

In the full/full reuse regime, $(\Lambda_1^*, \Lambda_2^*) = (N_1, N_2)$. In the full/partial reuse regime, the sparser network sets $\Lambda_1^* = N_1$ and the denser network sets Λ_2^* as the solution to (8). In the partial/partial reuse regime, $(\Lambda_1^*, \Lambda_2^*) = (\sqrt{\Lambda^*(\alpha/2)}, \sqrt{\Lambda^*(\alpha/2)})$.

The regimes are essentially distinguished by which boundary constraints are active. For $2 < \alpha < 4$, the partial/partial reuse regime is not accessible. Fig. 3 provides an illustrated means for determining which regime the system is in for a range of values of the path loss exponent. In these plots, we consider all values

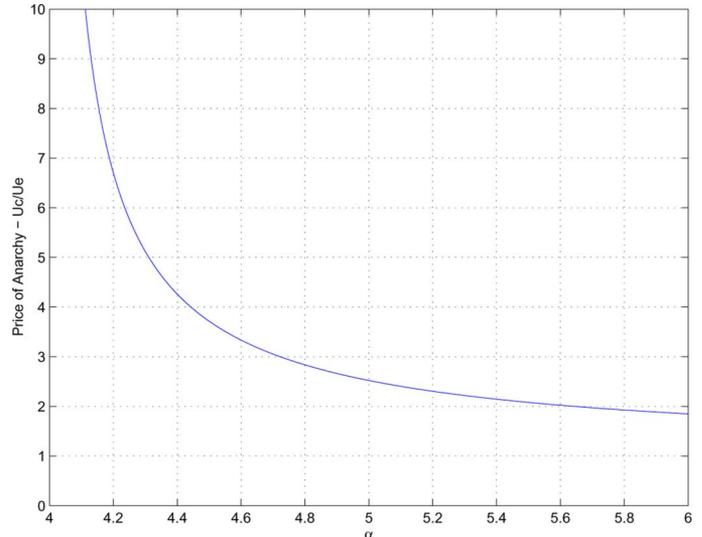


Fig. 4. For $\alpha > 4$, the price of anarchy depends only on the path loss exponent in the partial/partial reuse regime.

of N_1 and N_2 , not just those satisfying $N_1 \geq N_2$. Notice that as $\alpha \rightarrow 2$, the entire region corresponds to the full/full reuse regime; for $\alpha = 4$, almost the entire region corresponds to the full/partial reuse regime; and for $\alpha \rightarrow \infty$, the entire region corresponds to the partial/partial reuse regime.

B. Variable-Rate Model

In this section, we examine the case where tx-rx pairs tailor their communication rates to suit instantaneous channel conditions, sending at rate $\log(1 + \text{SIR}(t))$ during the t th time slot. Various protocols can be used to enable tx-rx pairs to estimate their $\text{SIR}(t)$.

Consider first the single, isolated network scenario. The expected time-averaged rate per user is now

$$\mathbb{E}\bar{R}_i = p_i \mathbb{E} \log(1 + \text{SIR}).$$

As before, the rate is both time-averaged over the interference and averaged over the geographic distribution of the nodes. The SIR is the instantaneous value observed by a given rx node and is distributed according to

$$\mathbb{P}(\text{SIR} > x) = \mathbb{P}(r > x^{1/\alpha}d) = \left(1 - \frac{\pi x^{2/\alpha} d^2 p_i}{n}\right)^{n\lambda_i - 1}$$

where the variable r denotes the distance to the nearest interferer. By optimizing over Λ_i one arrives at

$$\mathbb{E}\bar{R}_i \rightarrow \frac{1}{N_i} \max_{0 \leq \Lambda_i \leq N_i} \Lambda_i \int_0^\infty e^{-\Lambda_i x^{2/\alpha}} \frac{dx}{1+x}. \quad (9)$$

Define $\Lambda'(\alpha)$ as the maximizing argument of the unconstrained version of the above optimization problem or, more specifically, as the unique solution to

$$\int_0^\infty \frac{1 - \Lambda' x^{2/\alpha}}{1+x} e^{-\Lambda' x^{2/\alpha}} dx = 0.$$

The function $\Lambda'(\alpha)$ is plotted in Fig. 6. Then, $p_i^* = \min(\Lambda'(\alpha)/N_i, 1)$. From this, we see that the solution for

the variable-rate case is the same as the fixed-rate solution, differing only by substitution of the function $\Lambda'(\alpha)$ for $\Lambda^*(\alpha)$.

Now, we turn to the case of two competing wireless networks. Using an approach similar to the previous one, it can be shown that

$$\mathbb{E}\bar{R}_i \rightarrow \frac{1}{N_i} \Lambda_i \int_0^\infty e^{-(\Lambda_1 + \Lambda_2)x^{2/\alpha}} \frac{dx}{1+x}.$$

In this way, we can define the game between the two networks like so.

Definition III-8: (Variable-Rate Random Access Game) A strategy for network i in the Variable-Rate Random Access Game is a choice of $\Lambda_i \in [0, \pi\lambda_i d^2]$. The payoff functions are

$$U_1(\Lambda_1, \Lambda_2) = \Lambda_1 \int_0^\infty e^{-(\Lambda_1 + \Lambda_2)x^{2/\alpha}} \frac{dx}{1+x}$$

$$U_2(\Lambda_1, \Lambda_2) = \Lambda_2 \int_0^\infty e^{-(\Lambda_1 + \Lambda_2)x^{2/\alpha}} \frac{dx}{1+x}.$$

From the above definition, we see that the Fixed-Rate Game is derived from the Variable-Rate Game by merely applying a step-function lower bound to the players' utility functions, with the width of the step-function optimized, i.e.,

$$\int_0^\infty e^{-(\Lambda_1 + \Lambda_2)x^{2/\alpha}} \frac{dx}{1+x} \geq \max_{\beta_i > 0} e^{-(\Lambda_1 + \Lambda_2)\beta_i^{2/\alpha}} \int_0^{\beta_i} \frac{dx}{1+x}$$

$$= \max_{\beta_i > 0} \log(1 + \beta_i) e^{-(\Lambda_1 + \Lambda_2)\beta_i^{2/\alpha}}.$$

It is natural to wonder whether the N.E. of the Variable-Rate Game bears any relationship to the N.E. of the Fixed-Rate Game.

As in the Fixed-Rate Game, the utility functions of the Variable-Rate Game can be explicitly evaluated when Λ_1 and Λ_2 are large, yielding

$$U_1(\Lambda_1, \Lambda_2) \approx \frac{\Lambda_1}{(\Lambda_1 + \Lambda_2)^{\alpha/2}} \cdot \Gamma(\alpha/2 + 1)$$

$$U_2(\Lambda_1, \Lambda_2) \approx \frac{\Lambda_2}{(\Lambda_1 + \Lambda_2)^{\alpha/2}} \cdot \Gamma(\alpha/2 + 1).$$

Compared to (4) and (5), we see that for large Λ_1, Λ_2 , the utility functions of the Variable-Rate Game have exactly the same functional dependency on Λ_1, Λ_2 as the utility functions of the Fixed-Rate Game, differing only in an α -dependent constant.

As anticipated by the above discussion, the N.E. behavior of the Variable-Rate Game parallels that of the Fixed-Rate Game. The same two modes are present, $2 < \alpha < 4$ and $\alpha > 4$. These give rise to the same three spreading regimes, the only difference being that the boundaries delineating them are shifted slightly. The N.E. values $(\Lambda_1^*, \Lambda_2^*)$ in each regime take on a similar form.

Theorem III-9: The Variable-Rate Random Access Game has a unique N.E. $(\Lambda_1^*, \Lambda_2^*)$, which lies in one of three regions. Let $\Lambda''(\alpha)$ be the unique solution of

$$\int_0^\infty \frac{1 - \Lambda'' x^{2/\alpha}}{1+x} e^{-2\Lambda'' x^{2/\alpha}} dx = 0 \quad (10)$$

for $\alpha > 4$ and equal to positive infinity for $\alpha \leq 4$.

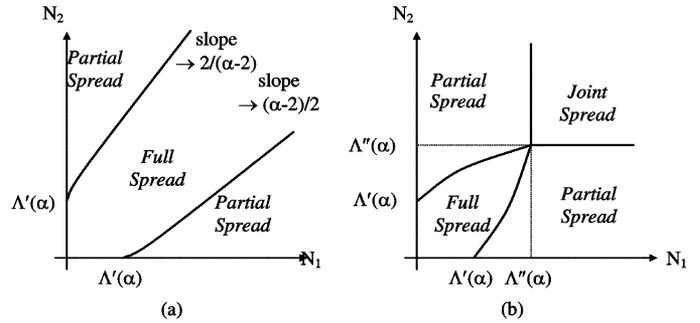


Fig. 5. The three regimes for the N.E. of the Variable-Rate Game.

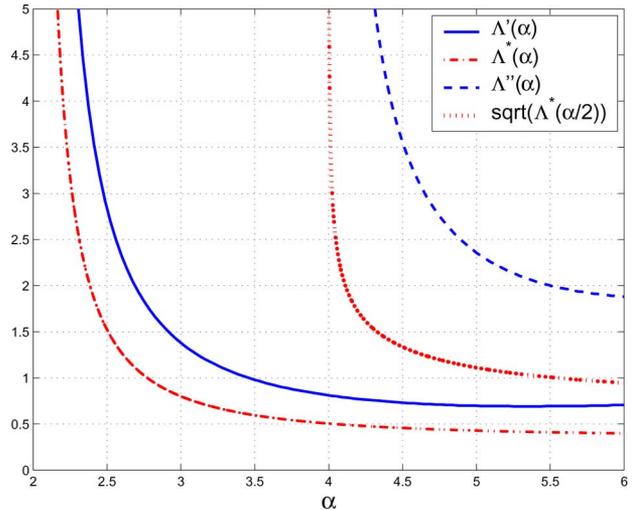


Fig. 6. The functions $\Lambda''(\alpha)$ and $\Lambda'(\alpha)$ for the Variable-Rate model versus the equivalent functions $\Lambda^*(\alpha)$ and $\sqrt{\Lambda^*(\alpha/2)}$ for the Fixed-Rate model.

- (Full/Full reuse) If $N_1 \leq \Lambda''(\alpha)$ and $N_2 \leq$, the unique solution over Λ of

$$\int_0^\infty \frac{1 - \Lambda x^{2/\alpha}}{1+x} e^{-(N_1 + \Lambda)x^{2/\alpha}} dx = 0 \quad (11)$$

then $(\Lambda_1^*, \Lambda_2^*) = (N_1, N_2)$.

- (Full/Partial reuse) If $N_1 \leq \Lambda''(\alpha)$ and $N_2 >$ the unique solution of (11), then $\Lambda_1^* = 1$ and Λ_2^* is equal to this unique solution.
- (Partial/Partial reuse) If $N_1 > \Lambda''(\alpha)$, then $(\Lambda_1^*, \Lambda_2^*) = (\Lambda''(\alpha), \Lambda''(\alpha))$.

A regime map is provided in Fig. 5. As is evident from the above theorem, it is not possible to characterize the behavior of the N.E. for the Variable-Rate Game as explicitly as for the Fixed-Rate Game. This is in part due to the more complex representation of the utility functions in terms of integrals and in part due to the fact that the function $\Lambda''(\alpha)$ cannot be represented in terms of the function $\Lambda'(\alpha)$, as in the case of the Fixed-Rate Game where one function equals the square root of the other evaluated at $\alpha/2$.

For large N_1 , however, we can use the approximation adopted in Theorem III-5 to explicitly characterize the behavior of the N.E. in the full/partial reuse regime.

Theorem III-10: (Variable-Rate Random Access N.E. for $2 < \alpha < 4$ and $N_1 \gg 1$) In the limit $N_1 \rightarrow \infty$, the N.E. occurs at

$$(\Lambda_1^*, \Lambda_2^*) = \begin{cases} (N_1, N_2), & N_1 \leq N_2 \leq \frac{2}{\alpha-2}N_1 \\ (N_1, \frac{2}{\alpha-2}N_1), & \frac{2}{\alpha-2}N_1 \leq N_2. \end{cases}$$

Thus for $2 < \alpha < 4$ and large N_1 , the behavior of the N.E. in the Variable-Rate Game is identical to that of the Fixed-Rate Game. As discussed earlier, the values of U_1 and U_2 at equilibrium are equal to those of the Fixed-Rate Game times a constant $(\alpha/2)^{\alpha/2}e^{-\alpha/2}/\Gamma(\alpha/2 + 1)$.

C. Explanation of Behavior

The intuition behind our result is the following. The average throughput per link is essentially equal to the product of the fraction of time transmissions are scheduled, and the average number of bits successfully communicated per transmission. Adjusting the transmit density has a linear effect on the former term, but a nonlinear effect on the latter. The latter depends on the SIR, and the SIR essentially depends on the path loss exponent via

$$\text{SIR} \approx \left(\frac{\text{distance to transmitter}}{\text{distance to interferer}} \right)^\alpha.$$

When the nearest interferer is closer than the transmitter, the ratio inside the parentheses is less than 1, and a large value of α substantially hurts the SIR, dragging it to near zero and causing the link capacity to drop to near zero. However, when the nearest interferer is further away than the transmitter, the ratio is greater than 1 and a large value of α substantially improves the SIR, resulting in a large link capacity. Thus for large α , the average number of bits successfully communicated per transmission is very sensitive to whether or not the transmission disc is empty.

This insensitivity for sufficiently small α means that increasing the transmit density in network i causes a linear increase in the fraction of time transmissions are scheduled, but has little effect on the number of bits successfully communicated per transmission, up until the point where the transmit density of network i starts to dwarf the transmit density of network j . Thus, network i will wish to increase its transmit density until it is sufficiently larger than network j 's. Likewise, network j will wish to increase its transmit density until it is sufficiently larger than network i 's. Ultimately this results in either:

- 1) a full/full reuse solution, which occurs when the sparser network maxes out and winds up simultaneously scheduling all of its transmissions, and the denser network is insufficiently dense such that its optimal transmit density based on the sparser networks choice results in it simultaneously scheduling all of its transmissions; or
- 2) a full/partial reuse solution, which occurs when the sparser network maxes out and winds up simultaneously scheduling all of its transmissions, but the denser network is sufficiently dense such that its optimal transmit density based

on the sparser networks choice results in it simultaneously scheduling only a fraction of its transmissions.

The opposite effect occurs for sufficiently large α , where the average number of bits successfully communicated per transmission depends critically on whether or not there is an interferer inside the transmission disc. In this scenario, network i will set its active density to a level lower than network j 's in order to capitalize on those instances in which network j happens to not schedule any transmissions nearby to one of network i 's receivers, resulting in the successful communication of a large number of bits. Likewise, network j will set its active density to a level lower than network i 's, and the system converges to the partial/partial reuse regime.

IV. VARIABLE TRANSMISSION RANGE

One of our initial assumptions was that all tx-rx pairs have the same transmission range d . In this section, we consider the scenario where the transmission ranges of all tx-rx pairs in the system are i.i.d. random variables D_j . When the variance of D_j is large, some form of power control may be required to ensure long-range transmissions are not unfairly penalized. A natural form of power control involves tx nodes setting their transmit powers such that all transmissions are received at the same SNR. This means transmit power scales proportional to D_j^α . Denote the distance from the k th tx node to the j th rx node D_{kj} . Then, the interference power from the k th tx node impinging on the j th rx node is proportional to $D_{kk}^\alpha/D_{kj}^\alpha$. In the fixed transmission range scenario, this quantity was proportional to $1/D_{kj}^\alpha$. There we assumed the bulk of the interference was generated by the dominant interferer. Denote the scheduled set of tx nodes at time t by $\mathcal{A}(t)$. This assumption essentially evoked the following approximation:

$$\sum_{k \in \mathcal{A}(t)} 1/D_{kj}(t)^\alpha \approx \max_{k \in \mathcal{A}(t)} 1/D_{kj}(t)^\alpha.$$

The equivalent approximation for the variable transmission range problem is

$$\sum_{k \in \mathcal{A}(t)} D_{kk}^\alpha/D_{kj}(t)^\alpha \approx \max_{k \in \mathcal{A}(t)} D_{kk}^\alpha/D_{kj}(t)^\alpha.$$

Thus for variable range transmission, the dominant interferer is not necessarily the nearest to the receiver. Under this assumption, the SIR at the j th rx node at time t is then

$$\text{SIR}_j(t) = \frac{D_{k^*k^*}^{-\alpha}}{D_{k^*j^*}^{-\alpha}}$$

where k^* is the index of the tx node that is closest to the j th receiver relative to its transmission range.

Let us compute the throughput for the variable transmission range model under the Fixed-Rate Random Access protocol.

$$\mathbb{E}\bar{R} = p_i \mathbb{P}(\text{SIR}_j(t) > \beta_i) \log(1 + \beta_i).$$

The probability the SIR is greater than the threshold

$$\begin{aligned}
& \mathbb{P}(\text{SINR}_j(t) > \beta_i) \\
&= \mathbb{P}\left(D_{kj} > \beta_i^{1/\alpha} D_{kk}, \forall k\right) \\
&= \left(\int_0^{\sqrt{n/\pi}/\beta_i^{1/\alpha}} \mathbb{P}\left(D_{kj} > \beta_i^{1/\alpha} x\right) \mathbb{P}_{D_{kk}}(x) dx\right)^{n\lambda} \\
&= \left(1 - \frac{p_i \pi \beta_i^{2/\alpha} \mathbb{E}D_{kk}^2}{n}\right)^{n\lambda} \\
&\rightarrow e^{-p_i \pi \beta_i^{2/\alpha} \mathbb{E}D_{kk}^2}
\end{aligned}$$

as $n \rightarrow \infty$. For notational simplicity, let $d \equiv D_{kk}$. If we define N_i as the average number of nodes per transmission disc, where the average is taken over both the geographical distribution of the nodes and the distribution of the size of the transmission disc, i.e., $N_i = \pi \lambda_i \mathbb{E}d^2$, we wind up with $\mathbb{E}\bar{R} \rightarrow p_i \log(1 + \beta_i) e^{-N_i p_i \beta_i^{2/\alpha}}$, which is the same result as the fixed-transmission range model. It is straightforward to extend the analysis to the case of two competing networks. The throughput per user in network 1 is then

$$\mathbb{E}\bar{R} \rightarrow p_1 \log(1 + \beta_1) e^{-(N_1 p_1 + N_2 p_2) \beta_1^{2/\alpha}}$$

and likewise for network 2. From this, we see that all results for the fixed-transmission range model extend to the variable-transmission range model by simply replacing d^2 by $\mathbb{E}d^2$.

V. SIMULATIONS

A. Random Access Protocol

In order to get a sense of the typical behavior of the players in the (Variable-Rate) Random Access Game, and to justify the validity of the Dominant Interferer assumption, we simulate the behavior of the following greedy algorithm with the interference computed based on all transmissions in the network, not just the strongest.

Inputs: $p_1(0), p_2(0), \Delta$

Outputs: $\mathbf{p}_i = [p_i(1), \dots, p_i(500)]$, for $i = 1, 2$.

For $t = 1$ to 500

Form estimate $\bar{R}_1(p_1(t-1) + \Delta, p_2(t-1))$

Form estimate $\bar{R}_1(p_1(t-1) - \Delta, p_2(t-1))$

If $\bar{R}_1(p_1(t-1) + \Delta, p_2(t-1)) > \bar{R}_1(p_1(t-1) - \Delta, p_2(t-1))$ $p_1(t) = \min(p_1(t-1) + \Delta, 1)$

Else

$p_1(t) = \max(p_1(t-1) - \Delta, 0)$

End

Form estimate $\bar{R}_2(p_1(t), p_2(t-1) + \Delta)$

Form estimate $\bar{R}_2(p_1(t), p_2(t-1) - \Delta)$

If $\bar{R}_2(p_1(t), p_2(t-1) + \Delta) > \bar{R}_2(p_1(t), p_2(t-1) - \Delta)$ $p_2(t) = \min(p_2(t-1) + \Delta, 1)$

Else

$p_2(t) = \max(p_2(t-1) - \Delta, 0)$

End

End

Each update time t , network 1 performs a simple gradient descent search by temporarily increasing and then decreasing its access probability to determine which option leads to a higher throughput. Shortly afterward, network 2 does the same. If Δ is small, then both networks can perform these measurement operations simultaneously without significantly affecting the outcome.

The topology used in the simulations consisted of 400 tx nodes from network 1 and 200 tx nodes from network 2, all i.i.d. uniformly distributed on the surface of a sphere of unit area. For each tx node, its corresponding rx node was located at a point randomly chosen at uniform from a disc of radius 0.15. This corresponds to $N_1 = 400/\sqrt{(800)} \approx 14.14$ and $N_2 = 200/\sqrt{(800)} \approx 7.28$. A step-size of $\Delta = 0.02$ was used. The results for $\alpha = 2.5, 3.5$, and 4.5 are displayed in Fig. 7. The observed behavior corresponds to the analytical results. For the values of N_1 and N_2 used, the N.E. lies in the Full/Full regime for $\alpha = 2.5$, the Full/Partial regime for $\alpha = 3.5$, and the Partial/Partial regime for $\alpha = 4.5$, as can be seen from Fig. 3.

B. Carrier-Sensing Multiple-Access-Based Protocol

The high-level conclusion from our analysis of the Random Access protocol is that the N.E. is cooperative in nature for a sufficiently high path loss exponent. Ideally, we would like to be able to draw this conclusion for a more sophisticated class of scheduling protocols employing carrier sensing. Due to the analytical intractability of the problem, we present simulation results to illustrate this effect. We assume both networks operate under the following protocol. We present a centralized version of it due to space constraints, but claim there exists a distributed version that performs identically in most cases. During the scheduling phase, each tx-rx pair is assigned a unique token at random from $\{1, \dots, n\lambda_i\}$. Tx nodes proceed with their transmission so long as they will not cause excessive interference to any rx node with a higher priority token. More precisely, a tx node x will be scheduled for transmission so long as it will not cause the SINR of any higher priority rx node to fall below a *silencing threshold* γ_i ($i = 1$ for network 1 and $i = 2$ for network 2). These higher priority SINR's are computed as the ratio of the received signal power to the interference of tx node x plus noise. Thus, a game between the two networks can be defined where the strategies are the choices of silencing thresholds γ_1 and γ_2 . We refer to this as the *CSMA Game*. The silencing threshold for the CSMA Game essentially plays the same role as the access probability in the Random Access Game: It determines the degree of spatial reuse. A high value of γ leads to a low density of transmissions, and a low value of γ leads to a high density.

We simulate the behavior that arises when both networks optimize their silencing thresholds in a greedy manner. Analogously to before, we have the following algorithm.

Inputs: $p_1(0), p_2(0), \Delta$

Outputs: $\mathbf{p}_i = [p_i(1), \dots, p_i(500)]$, for $i = 1, 2$.

For $t = 1$ to 500

Form estimate $\bar{R}_1(\gamma_1(t-1) + \Delta, \gamma_2(t-1))$

Form estimate $\bar{R}_1(\gamma_1(t-1) - \Delta, \gamma_2(t-1))$

If $\bar{R}_1(\gamma_1(t-1) + \Delta, \gamma_2(t-1)) > \bar{R}_1(\gamma_1(t-1) - \Delta, \gamma_2(t-1))$

```

 $\gamma_1(t) = \min(\gamma_1(t-1) + \Delta, 30 \text{ dB})$ 
Else
 $\gamma_1(t) = \max(\gamma_1(t-1) - \Delta, -30 \text{ dB})$ 
End
Form estimate  $\bar{R}_2(\gamma_1(t), \gamma_2(t-1) + \Delta)$ 
Form estimate  $\bar{R}_2(\gamma_1(t), \gamma_2(t-1) - \Delta)$ 
If  $\bar{R}_2(\gamma_1(t), \gamma_2(t-1) + \Delta) > \bar{R}_2(\gamma_1(t), \gamma_2(t-1) - \Delta)$ 
 $\gamma_2(t) = \min(\gamma_2(t-1) + \Delta, 30 \text{ dB})$ 
Else
 $\gamma_2(t) = \max(\gamma_2(t-1) - \Delta, -30 \text{ dB})$ 
End
End

```

The topology used in the setup is identical to before, the only exception being that at each iteration of the algorithm, 10 old tx-rx pairs leave each network, and 10 new pairs join in i.i.d. locations drawn uniformly at random. This is to ensure sufficient averaging.

In a similar fashion to before, each network estimates the effect of either increasing or decreasing the silencing threshold, and then makes a permanent choice. For the same parameter values, the results of the simulation are displayed in Fig. 8. On the y-axes of these plots, we have drawn the fraction of nodes simultaneously scheduled at each iteration, which we denote f_1 and f_2 , rather than the silencing thresholds γ_i in order to draw a simple visual comparison with Fig. 7. For this reason, there is more fluctuation in the results, as the fraction of simultaneously scheduled transmissions varies not only due to the up/down movements of the silencing thresholds, but also due to the changing topology.

We conclude from these plots that for small values of α (namely $\alpha = 2.5$), the system converges to a competitive equilibrium, where both networks simultaneously schedule a large fraction of their transmissions, and for large values of α (namely $\alpha = 3.5$ and 4.5) the system converges to a near cooperative equilibrium, where both networks schedule a small fraction of their transmissions.

VI. CONCLUSION

This work studied spectrum sharing between wireless devices operating under a random access protocol. The crucial assumption made was that nodes belonging to the same network or coalition cooperate with one another. Competition only exists between nodes belonging to rival networks. It was found that cooperation between devices within the same network created the necessary incentive to prevent total anarchy between networks. For path loss exponents greater than 4, we showed that contrary to one's intuition, there can be a natural incentive for devices to cooperate to the extent that each occupies only a fraction of the available bandwidth. Such results are optimistic and encouraging. We demonstrated via simulations that it may be possible to extend them to more complex operating protocols such as those that employ carrier sensing to determine when the medium is free. More generally, one wonders whether a multistage game capturing the system dynamics under such a protocol can be formulated and whether the desirable properties of

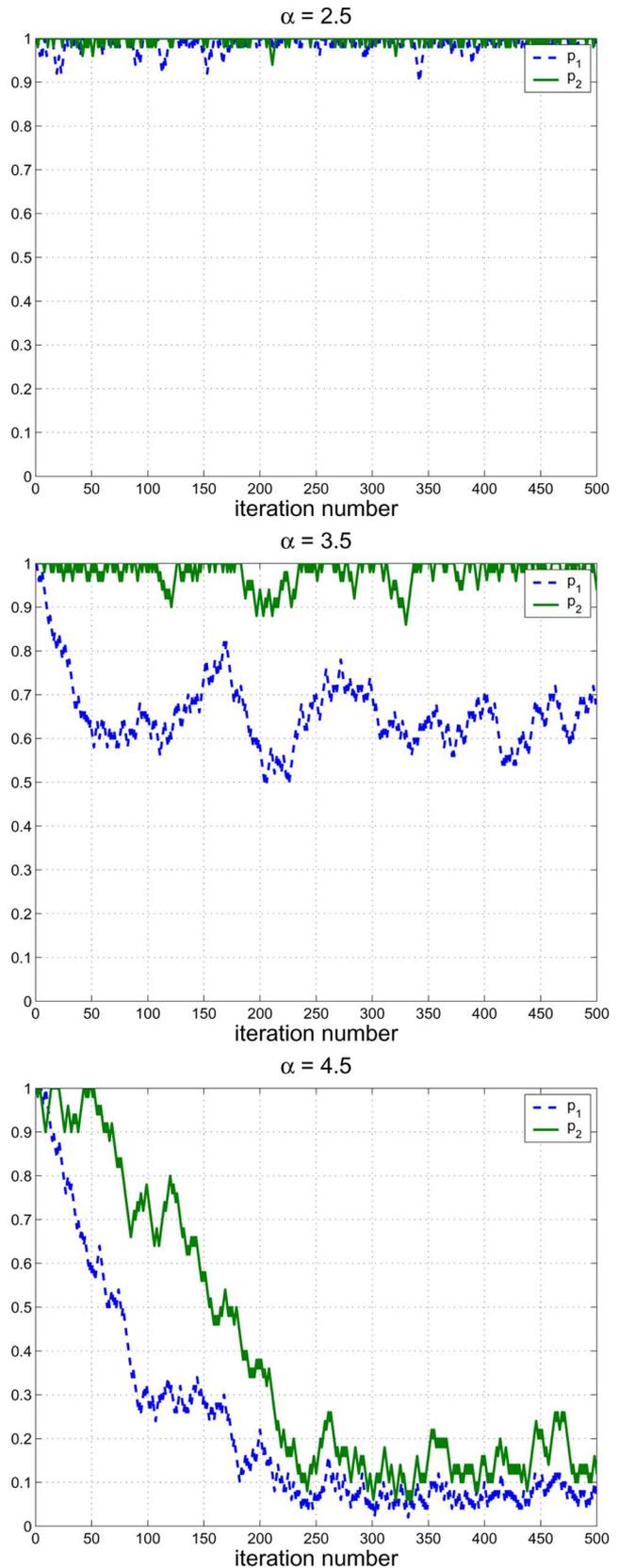


Fig. 7. Simulations of greedy algorithm under Random Access protocol.

the single-stage game continue to hold. It would also be worthwhile investigating the incentives wireless links have to form

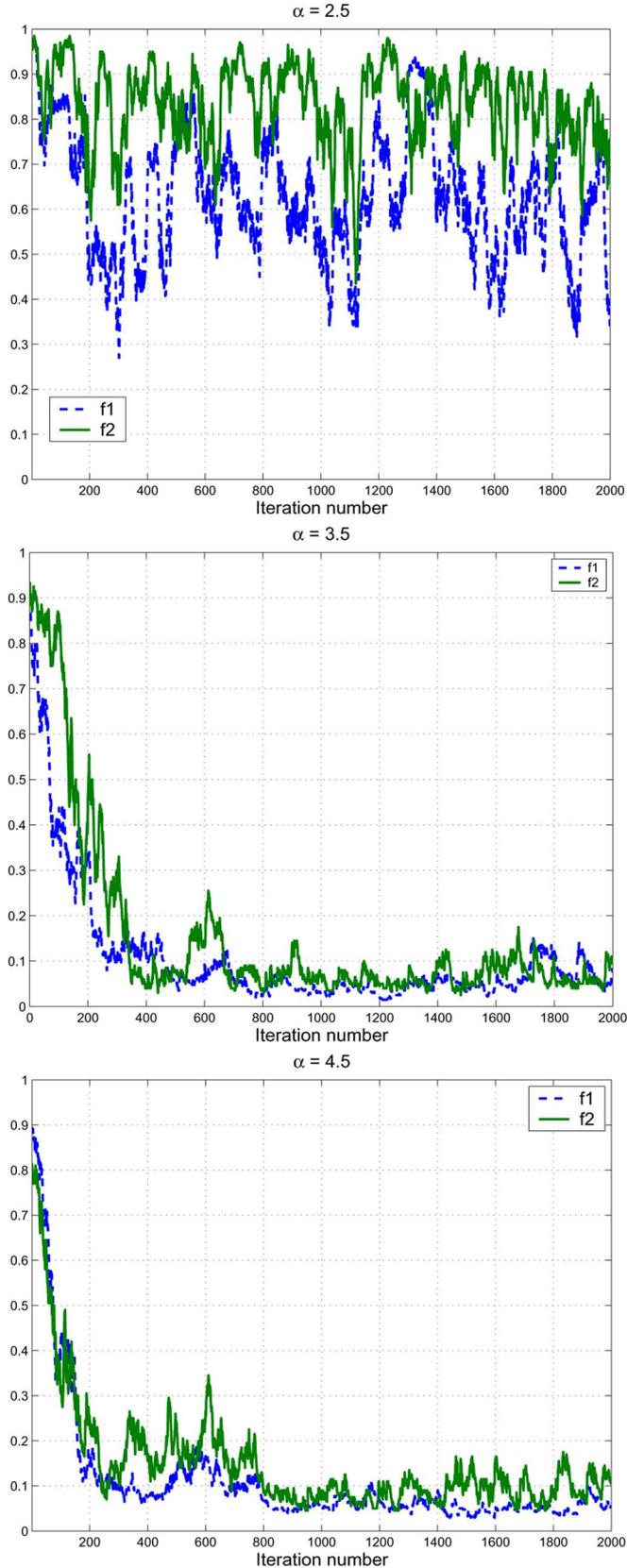


Fig. 8. Simulations of greedy algorithm under Carrier-Sensing Multiple Access protocol.

coalitions, as it was in essence assumed in this work that coalitions had been predetermined. Such an investigation would re-

quire extending the results of this work to the case of more than two competing networks.

APPENDIX PROOFS

For brevity, we only include the proof of Theorem III-7. The proof of Theorem III-4 is subsumed by that of III-7. The proof of Theorem III-9 is more or less built upon the same ideas as that of III-7. The remaining proofs are straightforward.

Theorem III-7: First consider the partial/partial reuse regime where $N_1 > \sqrt{\Lambda^*(\alpha/2)}$. We show that a N.E. cannot occur on the boundary of the strategy space. Suppose $\Lambda_1^* = N_1$. Then, $\Lambda_1^* > \sqrt{\Lambda^*(\alpha/2)}$. As $\sqrt{\Lambda^*(\alpha/2)}$ is the solution to the equation

$$\frac{\alpha}{2 \left(1 + \sqrt{\Lambda^*(\alpha/2)}^{\alpha/2}\right) \log \left(1 + \sqrt{\Lambda^*(\alpha/2)}^{-\alpha/2}\right)} - 1 = 1$$

and $(1+x) \log(1+1/x)$ is a monotonically decreasing function for $x > 0$, we have

$$\frac{\alpha}{2 \left(1 + \Lambda_1^{*\alpha/2}\right) \log \left(1 + \Lambda_1^{*-\alpha/2}\right)} - 1 > 1.$$

If Λ_1^* lies on the boundary of the strategy space, then $\partial U_1(\Lambda_1^*, \Lambda_2^*) / \partial \Lambda_1 > 0$, which implies

$$\begin{aligned} \Lambda_2^* &> \Lambda_1^* \left(\frac{\alpha}{2 \left(1 + \Lambda_1^{*\alpha/2}\right) \log \left(1 + \Lambda_1^{*-\alpha/2}\right)} - 1 \right) \\ &> \Lambda_1^* > \sqrt{\Lambda^*(\alpha/2)}. \end{aligned}$$

This in turn implies

$$\frac{\alpha}{2 \left(1 + \Lambda_2^{*\alpha/2}\right) \log \left(1 + \Lambda_2^{*-\alpha/2}\right)} - 1 > 1.$$

At equilibrium, $\partial U_2(\Lambda_1^*, \Lambda_2) / \partial \Lambda_2 \geq 0$, so

$$\Lambda_1^* \geq \Lambda_2^* \left(\frac{\alpha}{2 \left(1 + \Lambda_2^{*\alpha/2}\right) \log \left(1 + \Lambda_2^{*-\alpha/2}\right)} - 1 \right) > \Lambda_2^*$$

a contradiction. Thus, $\Lambda_1 < N_1$. Now assume $\Lambda_2 = N_2$. By assumption, $N_2 \geq N_1$, so $\Lambda_2^* > \sqrt{\Lambda^*(\alpha/2)}$. By repeating the same arguments, we can generate the same style of contradiction, and thus $\Lambda_2 < N_2$. This establishes that a N.E. cannot occur on the boundary of the strategy space. In essence, each network is trying to undercut the active density of the other. This drags the equilibrium away from the boundary.

Now, we establish any N.E. must be symmetric, i.e., $\Lambda_1^* = \Lambda_2^*$. Suppose a N.E. with $\Lambda_1^* \neq \Lambda_2^*$ exists. Then, as it must lie on the interior of the strategy space and as the utility functions are symmetric, $(\Lambda_2^*, \Lambda_1^*)$ must also be a N.E. On the interior of the strategy space, the N.E. criterion is $\partial U_1(\Lambda_1^*, \Lambda_2^*) / \partial \Lambda_1 = 0$, and so the function $\Lambda_1^*(\Lambda_2)$ is monotonically increasing in Λ_2 .

However, this implies we cannot have N.E. at both $(\Lambda_1^*, \Lambda_2^*)$ and $(\Lambda_2^*, \Lambda_1^*)$, a contradiction. Thus, $\Lambda_1^* = \Lambda_2^*$.

By differentiating the utility functions, this implies that at any N.E., Λ_1^* satisfies

$$\frac{\alpha}{2 \left(1 + \Lambda_1^{*\alpha/2}\right) \log \left(1 + \Lambda_1^{*-\alpha/2}\right)} - 1 = 1$$

with $\Lambda_2^* = \Lambda_1^*$. However, this is equivalent to $\Lambda_1^* = \sqrt{\Lambda^*(\alpha/2)}$. Thus, the N.E. is unique and occurs at $(\Lambda_1^*, \Lambda_2^*) = \left(\sqrt{\Lambda^*(\alpha/2)}, \sqrt{\Lambda^*(\alpha/2)}\right)$.

Next, consider the partial spread and full spread regimes where $N_1 \leq \sqrt{\Lambda^*(\alpha/2)}$. We first show that $\Lambda_1^* = N_1$. Suppose $\Lambda_1^* < N_1$. Then, $\Lambda_1^* < \sqrt{\Lambda^*(\alpha/2)}$, which implies

$$\frac{\alpha}{2 \left(1 + \Lambda_1^{*\alpha/2}\right) \log \left(1 + \Lambda_1^{*-\alpha/2}\right)} - 1 < 1.$$

At equilibrium, $\partial U_1 / \partial \Lambda_1 = 0$, so

$$\begin{aligned} \Lambda_2^* &= \Lambda_1^* \left(\frac{\alpha}{2 \left(1 + \Lambda_1^{*\alpha/2}\right) \log \left(1 + \Lambda_1^{*-\alpha/2}\right)} - 1 \right) \\ &< \Lambda_1^* < \sqrt{\Lambda^*(\alpha/2)}. \\ &\Rightarrow \frac{\alpha}{2 \left(1 + \Lambda_2^{*\alpha/2}\right) \log \left(1 + \Lambda_2^{*-\alpha/2}\right)} - 1 < 1 \end{aligned}$$

which, in conjunction with the equilibrium condition $\partial U_2 / \partial \Lambda_2 = 0$, implies $\Lambda_1^* < \Lambda_2^*$, a contradiction. Thus, $\Lambda_1^* = N_1$. Now, we can solve for Λ_2^* to conclude that Λ_2^* is the unique solution to (8) or N_2 , whichever is smaller.

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