Research Note

The Superposition of Sine Waves of Equal Frequency

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Abstract

It is well known that the superposition (sum) of any number of pure sinusoidal waveforms of equal frequency but arbitrary relative amplitudes and phases is in general another pure sinusoid of the same frequency whose amplitude and phase depends on the specific amplitudes and phases of the components. This is the reason why, for example, when a chorus of singers together sing the same note, we can hear that note clearly regardless of the arbitrary mismatches in amplitude and phase that may exist between the individual voices. The different voices generally tend to reinforce each other, rather than canceling each other out; complete cancellation would occur only in a measure-zero subset of the possible cases. In this note, we derive from first principles the basic trigonometric identities that form the quantitative foundations of this useful fact.

1. Sine Plus Cosine

First, we will derive a trigonometric identity for the simple case

$$a \cos x + \sin x = b \sin(x + \theta) \tag{1}$$

where all variables are real, and where b and θ are expressed explicitly as closed-form functions of a. This corresponds to the special case where the two component sinusoids are exactly 90 degrees out of phase. We will solve this problem by solving the more general complex equation

$$a\, e^{i(x + \pi/2)} + e^{ix} = b\, e^{i(x + \theta)} \tag{2}$$

of which eq. (1) is merely the real part. We start by factoring the exponentials:

$$a\, e^{ix} e^{i\pi/2} + e^{ix} = b\, e^{ix} e^{i\theta} \tag{3}$$

Now, e^{ix} divides out, giving us

$$ia + 1 = b\, e^{i\theta}. \tag{4}$$

This equation already makes it obvious that

$$\theta = \operatorname{atan}(a), \qquad b = (a^2 + 1)^{1/2}. \tag{5}$$

Thus, the desired identity is

$$a \cos x + \sin x = (a^2 + 1)^{1/2} \sin(x + \operatorname{atan} a). \quad (6)$$

We can thus see that the superposition of the sine and cosine is a new sinusoidal waveform of the same frequency and amplitude greater than either component alone. The phase shift relative to the single sin x voice is given by the arctangent of a, the relative amplitude of the cosine and sine components.

2. Sine Plus Phase-Shifted Sine

In this section, we generalize the above to the more general case

$$a \sin(x + \varphi) + \sin x = b \sin(x + \theta), \quad (7)$$

where now φ may be any arbitrary phase difference (not necessarily $\pi/2$) between the component voices. Eq. 2 generalizes to

$$a\, e^{i(x + \varphi)} + e^{ix} = b\, e^{i(x + \theta)} \quad (8)$$

and eq. 4 becomes

$$a e^{i\varphi} + 1 = b\, e^{i\theta}. \quad (9)$$

To solve for b and θ requires transforming the left-hand side into phase-magnitude form. Expanding the exponential using Euler's relation and gathering the reals and then transforming back to phase-magnitude gives

$$\begin{aligned} & a(\cos \varphi + i \sin \varphi) + 1 \\ &= (a \cos \varphi + 1) + i(\sin \varphi) \\ &= [(a \cos \varphi + 1)^2 + \sin^2 \varphi]\exp[i \cdot \operatorname{atan2}(a \cos \varphi + 1, \sin \varphi)], \end{aligned} \quad (10)$$

where atan2 is the four-quadrant extension of atan where the result may be an angle in any of the four quadrants, that is, anywhere in $[-\pi,\pi)$. So,

$$\begin{aligned} b &= (a \cos \varphi + 1)^2 + \sin^2 \varphi, \\ \theta &= \operatorname{atan2}(a \cos \varphi + 1, \sin \varphi) \end{aligned} \quad (11)$$

and so the general formula that we seek is

$$a \sin(x + \varphi) + \sin x = [(a \cos \varphi + 1)^2 + \sin^2 \varphi] \sin[x + \operatorname{atan2}(a \cos \varphi + 1, \sin \varphi)]. \quad (12)$$

Note that the two voices can only cancel out entirely if $b = 0$. Since both terms in b are quadratic and thus nonnegative, we can only have $b = 0$ when both

$$\sin \varphi = 0 \quad (13)$$

and

$$a \cos \varphi + 1 = 0. \quad (14)$$

These two equations can be seen to imply that $ae^{i\varphi} = -1$, so that the second waveform $a \sin(x+\varphi)$ is always exactly the negative of the first, as desired. Note that this only happens when $\varphi = n\pi$, with n being exactly an integer, and when $a = -\cos\varphi$; thus, only in a vanishingly small fraction of possible cases. In general, multiple sinusoidal voices of the same frequency do not cancel, but instead produce a combined sinusoidal voice of that same frequency, with an amplitude and phase that depend on the relative amplitude and phase of the components as according to eq. 12.

3. Summary

In the above, we derived the following general trigonometric identity:

$$a \sin(x + \varphi) + \sin x = [(a \cos\varphi + 1)^2 + \sin^2\varphi] \sin[x + \operatorname{atan2}(a \cos\varphi + 1, \sin\varphi)].$$

in the special case where $\varphi = \pi/2$, this can be simplified to:

$$a \cos x + \sin x = (a^2 + 1)^{1/2} \sin(x + \operatorname{atan} a).$$