The Indefinite Logarithm, Logarithmic Units, and the Nature of Entropy

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Abstract

We define the *indefinite logarithm* $[\log x]$ of a real number $x > 0$ to be a mathematical object representing the abstract concept of the logarithm of x with an *indeterminate* base (*i.e.*, not specifically e, 10, 2, or any fixed number). These indefinite logarithmic quantities naturally play a mathematical role that is closely analogous to that of dimensional physical quantities (such as length) in that, although these quantities have no definite interpretation as ordinary numbers, nevertheless the *ratio* of two of these entities is naturally well-defined as a specific, ordinary number, just like the ratio of two lengths. As a result, indefinite logarithm objects can serve as the basis for *logarithmic spaces*, which are natural systems of logarithm units suitable for measuring any quantity defined on a logarithmic scale. Logarithmic units provide a convenient language for explaining the complete conceptual unification of the disparate systems of units presently used for quantities that are conventionally considered distinct, such as physical entropy and information-theoretic entropy.

Disclaimer: The ideas in this article seem to be implicitly understood as standard background knowledge in certain circles, and most of them would probably be judged to be exceedingly elementary and obvious among most professional mathematicians. These concepts are, most likely, already described (although likely with different terminology) in many places in the mathematics and physics literature. Thus, the purpose of this document is not to claim any sort of priority, but rather simply to review and explain these concepts, as well to solicit from readers appropriate references to prior work in this area.

1 Introduction

The motivation for this paper is to help clear up what is perceived to be a widespread confusion that can found in many popular sources (websites, popular books, *etc.*) regarding the proper mathematical status of a variety of physical quantities that are conventionally defined on logarithmic scales.

As an example, we focus on the quantity of thermodynamic entropy, and its close relationship to (and really, identity with) the concepts of entropy and information as defined within the context of information theory. Although many physicists and information theorists have understood quite well the mathematical reasons for the underlying unity between the entropy concepts in these two domains, many others still do not, and continue to believe that the physical and information-theoretic concepts of entropy are fundamentally different.

Some (although not all) of the confusion that we have seen expressed in this regard can be traced back to the historical accident that thermodynamic entropy is most often measured in natural logarithm units, while information-theoretic entropy is more frequently measured in units of the logarithm base 2 (*i.e.*, in bits), or some multiple thereof. But of course, the choice of the logarithm base in the definition of entropy is completely inessential, and amounts merely to a choice of one's unit of measurement, as Shannon himself pointed out in his seminal work on information theory.

Even further, the supposed distinction between the "physical" nature of thermodynamic entropy (as measured in, say, Joules per Kelvin) and the allegedly more "mathematical" nature of information entropy (measured in bits) can also be seen as a totally artificial distinction, one resulting from nothing other than the fact that early thermodynamicists were not yet aware that physical entropy really *is* nothing other than a measurable manifestation of what is at root merely a purely mathematical, statistical quantity.

In fact, as we will review, any given unit of physical entropy can be *exactly* identified with a corresponding (purely abstract) *mathematical* unit, while still remaining consistent with all observed empirical data. The fundamental scientific principle of adopting the most parsimonious theory that explains the data (a.k.a. Ockham's razor) then *demands* that as good scientists we *must* indeed adopt this identification between the physical and mathematical domains, and take it seriously as holding the status of our best available model of reality, at least until empirical evidence to the contrary is found.

Although these issues are already quite well understood in certain circles, we nevertheless felt that, as a public service, it would be worthwhile to write a short paper that elaborates on the mathematical foundations of these issues in some detail. Two fundamental mathematical concepts which I have to be found rather useful in explaining these kinds of issues are concepts that I refer to as the "indefinite logarithm" and "logarithmic units." The definition and discussion of these concepts will form the main mathematical core of this paper.

Although this material seems to be already essentially common or intuitively obvious knowledge among many of the leading researchers who deal every day with the physics of information, I have found in my experience that misunderstandings and confusion regarding these issues nevertheless abound in other communities.

The reader should please note that, since this material seems to hold the status of being considered obvious or common knowledge in certain circles, this paper is by no means intended to claim any kind of intellectual priority on these issues. Rather, it is being written simply because the author is not presently aware of an accessible reference on this subject that explicitly explains these issues with a sufficient degree of pedagogical detail to satisfy general audiences.

The author welcomes comments and feedback from readers that may help point the author at seminal references or review articles in the mathematical literature that may elucidate these same issues, though quite possibly under different names.

2 The Indefinite Logarithm

We use the standard notation $\log_b a$ for the logarithm, base b, of a. Of course, this expression is well-defined for real $a, b > 0$, and even for all complex $a, b \neq 0$ as a multi-valued function.

But what if no particular base b is selected?

Of course, as a matter of notational convenience in mathematical literature $\log a$ is often defined to be simply a shorthand for the frequently-used natural logarithm, $\ln a = \log_e a$, or, in more everyday applied contexts, for the decimal-based $\log_{10} a$. But here, we are *not* asking about situations like these in which some definite base really exists but is merely left implicit by the notation. Rather, here we would like to discuss the concept of a 'new' kind of logarithm function wherein *no specific base is implied at all*. We dub this the *indefinite logarithm*, and we will give it a formal definition in a moment.

Of course, without a base, the result of the logarithm cannot be an ordinary number, since any specific number would imply that some specific base that was used. Instead, we can decare the output of the indefinite logarithm to be a different (non-numeric) type of mathematical object representing the result of performing this more abstract operation.

The form of this new type of object can be rigorously defined using standard mathematical concepts. Of course, as with any type of mathematical object, there is an infinite variety of ways in which we could satisfactorily represent these new objects in terms of more standard mathematical objects. Here is the representation that we find most convenient for purposes of this paper:

Definition 2.1 (Indefinite logarithm). For any given real number $x > 0$, the *indefinite logarithm L of x*, written $L = [\log x]$, is a special type of mathematical object called a "logarithmic quantity" object, which can be defined as follows:

$$L = [\log x] :\equiv \lambda(b > 0).\, \log_b x = \{(b, y) | b > 0,\ y = \log_b x\}. \tag{1}$$

Here, $:\equiv$ denotes "is defined as," and in the first expression after the $:\equiv$ we are using Church's lambda-calculus notation for functions, which gives us a concise way of saying that the indefinite logarithm $[\log x]$ for any given x is defined to be the unary function object $L : \mathbb{R} \to \mathbb{R}$ mapping real numbers $b > 0$ to the logarithm $y = \log_b x$ of the (given constant) x to the (variable, function argument) base b. Meanwhile, on the right, we are merely writing out the standard "graph representation" of this function object explicitly as the set of all ordered pairs (b, y) consisting of a base $b > 0$, followed by the ordinary (definite) logarithm $y = \log_b x$, of x to the base b.

Clearly, the result of the indefinite logarithm, as defined above, does not select any preferred base, yet it contains "all of the information" about the logarithm of x taken to all possible bases. So, in this sense it is really no less descriptive than a definite logarithm.

Although the above definition is restricted to positive real numbers (since this all that we need for our subsequent discussions), it could easily be extended to non-zero complex numbers if desired.

We can define addition of indefinite logarithms by adding their corresponding y values:

Definition 2.2 (Sum of indefinite logarithms). Given two indefinite logarithm objects L_1 and L_2, their sum $L = L_1 + L_2$ is defined by $L(b) :\equiv L_1(b) + L_2(b)$ for all $b > 0$. Or, stated a bit more formally using Lambda calculus, $L :\equiv \lambda b. L_1(b) + L_2(b)$.

We can similarly define negation of indefinite logarithms by simply negating their y values:

Definition 2.3 (Negation of indefinite logarithms). Given an indefinite logarithm object L, its negative $L' = -L$ is defined by $L'(b) :\equiv -L(b)$ for all $b > 0$. Or, $L' :\equiv \lambda b. -L(b)$.

If we add any indefinite logarithm to its negation, or take the indefinite logarithm of 1, we get a unique indefinite-logarithm object called the *null* or 0 indefinite logarithm, which returns 0 for all bases:

Definition 2.4 (Null indefinite logarithm). The indefinite logarithm of 1, *i.e.* $L_0 = [\log 1]$, is (by previous definitions) the function $\lambda b.0$ over reals $b > 0$. This L_0 will be called the *null indefinite logarithm* and will sometimes be written $[0]$.

Of course, $[0]$ is the identity element for the addition operation on natural logarithms; that is, for any L, we have $L + [0] = L$.

Note that there is *no* corresponding concept of a *unit* indefinite logarithm, *i.e.*, a multiplicative identity. That is, indefinite logarithms are inherently *scale-free* objects; that is, they are non-scalar quantities. Further, the space of indefinite logarithms does not even need to be considered to be closed under multiplication. A meaningful multiplication operation can be defined if a product of indefinite logarithms is considered to be a new type (similarly to how the product of two lengths is an area), but we will not develop that here.

Consistently with all of the above definitions, and with the ordinary definition of multiplication as repeated addition, indefinite logarithms can also be multiplied by arbitrary (positive, negative, or zero) real numbers:

Definition 2.5 (Indefinite logarithms multiplied by scalars). Given an indefinite logarithm quantity L and real number r, define the product $L' = rL = Lr$ of L times r by $L'(b) :\equiv r \cdot L(b)$. That is, $L' = \lambda b.rL(b)$.

Finally, solving the preceding expression for r allows us to recognize and define the result of the ratio of two indefinite logarithms as being an ordinary number:

Definition 2.6 (Ratio of indefinite logarithms). Given two indefinite logarithms L_1 and L_2, their ratio $r = L_1/L_2$ is defined as the real number $r :\equiv L_1(b)/L_2(b)$, where b is any positive real number. (The value of r does not depend on b.)

As an immediate consequence of the above definitions, the ratio of the indefinite logarithms of two numbers a and c is simply $[\log a]/[\log c] = \log_c a$, which (note) is the same as the ratio $\log_b a/\log_b c$ of the definite logarithms of a and c to any common base b. Thus, to emphasize, *the ratio of two logarithms is independent of what base we are working in, and thus remains well-defined even for indefinite logarithms*.

The above fact is important for our discussions in subsequent sections.

It is also worth noting that the indefinite logarithm shares all of the mathematically important properties of the ordinary logarithm (aside from not being a number), including the following useful identities:

- $[\log xy] = [\log x] + [\log y]$
- $[\log x/y] = [\log x] - [\log y]$
- $[\log x^y] = y[\log x]$.

Finally, it is useful to also define an *indefinite exponential* function, which maps a given indefinite logarithm object back to the unique real number of which it is the indefinite logarithm.

Definition 2.7 (Indefinite exponential). For any indefinite logarithm object $L = [\log x]$, let *the indefinite exponential of* L, written $[\exp L]$, be given by simply $[\exp L] = x$.

Another way to define $[\exp L]$, which is helpful when we are not explicitly given the x such that $L = [\log x]$, is simply to say that $[\exp L] = b^{L(b)}$, where $b > 0$ is any positive real number; all such b give the same value for $[\exp L]$.

3 Logarithmic Quantities

In the above, we occasionally referred to the indefinite logarithm objects as "quantities" in order to anticipate what we will now discuss, which is that indefinite logarithmic quantities (which we defined as pure mathematical entities) behave formally in a way that is exactly analogous to how dimensional *physical* quantities (such as length, time, and mass) behave. Indeed, logarithmic quantities can be viewed as "natural" dimensional quantities that exist independently of any particular models of physics.

Even further, later we will argue that logarithmic quantities can be understood as being the underlying essence behind certain quantities (in particular, thermodynamic entropy and physical information) that are frequently perceived as being "physical" rather than mathematical in nature. We will also argue that the question of whether these quantities are "really" physical or mathematical ones is an ill-posed one, being predicated on an entirely false dichotomy that has no real meaning.

First, what do we mean by a *quantity*, in general? (Regardless, for now, of whether it is supposed to be "mathematical" or "physical.") For our purposes, a quantity is an object selected from a set having a structure similar to that of the real number system, but without any built-in unit, that is, with no pre-ordained object to be designated "1." In abstract algebra terms, a set of quantities forms an (abstract) vector space over the reals, with a definite 0 quantity, an addition operation, a negation operation, and the ability to multiply by reals, but without a predefined unit quantity, and without necessarily any assigned meaning for a product of quantities.

A bit more generally and formally, we can define:

Definition 3.1 (Quantity spaces). Given any field F (in the standard abstract algebra sense of "field," *i.e.*, a commutative division ring with unity), a *quantity space Q over F* is simply a vector space over F, that is, an Abelian group of objects to be called *quantities*, which is closed under the additional operation of multiplication by the (scalar) elements of F.

Although quantity spaces, being vector spaces, may in general be many-dimensional, in this article we will primarily work with examples of quantity spaces that are only one-dimensional.

We can now define the concept of a *logarithmic (quantity) space*.

Definition 3.2 (Logarithmic spaces). A *logarithmic space* (or *logarithmic quantity space*) is a quantity space Q over $\mathbb{R}$ in which each quantity $q \in Q$ is identified with an indefinite logarithm object L (as defined in the preceding section), or with a series of indefinite logarithm objects, in the case of multidimensional logarithmic quantity spaces. The vector addition, negation, and scalar multiplication operations are identified with the corresponding operations on the indefinite logarithms. The null (0) vector comes from the null indefinite logarithm.

Now, a *logarithmic quantity* q is simply a member of some logarithmic quantity space Q. By a *scalar* logarithmic quantity or *logarithmic scalar*, we mean a member of a one-dimensional logarithmic space. Members of n-dimensional logarithmic quantity spaces will be called *n-dimensional* logarithmic quantities.

4 Logarithmic Units

Quantities in general (and logarithmic quantities in particular) have the property that there is no natural, built-in *unit* quantity that is automatically provided by the quantity space itself. However, in any given quantity space Q, we can always *choose* some arbitrary $u \in Q$ to be designated as a provisional *unit quantity*, and then all quantities in Q can be described in terms of scalar multiples of that unit. (For elements of multidimensional quantity spaces, a series of multiples is needed.) We may even have several different units $u_1, u_2, \ldots \in Q$, and express quantities sometimes as multiples of u_1, sometimes as multiples of u_2, *etc.*, and convert between expressions utilizing different units by multiplying them by appropriate conversion factors.

Of course, we are already familiar with these properties of quantities from their use in ordinary physics, in which (for example) spatial distances (and multi-dimensional displacement vectors) are considered to be quantities, rather than just pure numbers, and we can choose any number of units (meters, feet, *etc.*) for expressing them. Space itself (in the traditional continuum description) does not have any natural "unit length," only arbitrary units that we chose by convention.[1] Other examples of commonly used physical quantities include time, velocity, mass, and energy. (Of course, there are many others as well.)

Now, the primary observation of the previous section is that spaces of indefinite logarithm objects (logarithmic spaces) provide exactly the structure of quantity spaces; thus, we can represent all indefinite logarithm objects as scalar multiples of some arbitrarily chosen "unit" indefinite logarithm object. Indefinite logarithm objects thus naturally have the same mathematical status as do physical quantities.

5 Logarithmic Scales

Logarithmic quantities and units, in one form or another, are actually very widely used today, for quantifying a wide variety of things. Some examples include:

- Relative signal amplitudes or power levels, in physics and engineering.
- Earthquake strength (Richter scale) in seismology.

[1]Emerging theories of quantum gravity suggest that the Planck length $\ell_P = \sqrt{\hbar G/c^3}$ (or some small multiple of it) may play the role of a "natural" unit length, in some sense which is not yet fully understood. Nevertheless, we are still free, if we wish, to treat lengths as quantities that can be represented in arbitrary units.

- Tonal intervals on a musical scale.
- Entropy (in, as we will see, both the thermodynamic and information-theory senses).
- Information, in the information theory sense.

What is lacking presently, however, is the ubiquitous understanding among all communities that all of these disparate types of quantities can be understood as dealing with what is fundamentally the *same* underlying system of logarithmic units, as we defined above. The various "different" logarithmic scales that are in use are really distinguished only by a different choice of terminology for the logarithmic quantities and the units used for expressing them, and their application in describing different domains of study.

As examples, let us identify and name a variety of indefinite logarithm objects that are popularly used as units in which logarithmic quantities of interest are expressed in various fields. This list is sorted from smallest logarithmic unit to the largest, emphasizing that the units are comparable across domains.

- cent $= [\log 2]/1{,}200$. In music theory, the *cent* is 1/100th of a minor second, or $1/1{,}200^{\text{th}}$ of an octave.
- m2 $= [\log 2]/12$. In music theory, the *minor second* m2 is 100 cents or $1/12^{\text{th}}$ of an octave.
- M2 $= [\log 2]/6$. In music theory, the *major second* M2 is 200 cents or $1/6^{\text{th}}$ of an octave.
- dB $= 0.1[\log 10]$. The *decibel*. This is the smallest logarithmic unit in widespread use outside music theory, usually for expressing the magnitude of the ratio between signal strengths.
- b $= [\log 2]$. In information theory, the binary digit or *bit*. This is the smallest non-null logarithmic unit with an integer argument. In music theory, the same logarithmic unit is called an *octave* P8.
- n $= [\log e]$. The natural-log unit or *nat*. As we will explain in more detail later, this mathematical unit can be exactly identified with the physical unit k_{B} known as Boltzmann's constant. When used to express a ratio of current or voltage levels, the nat is called a *Neper* or Np.
- Np $= 2[\log e]$. The magnitude of the *Neper* of a ratio of currents or voltages, when translated to a ratio of power levels. (It is doubled because the power is the square of the current or voltage.)
- o $= [\log 8] = 3\,\mathrm{b}$. The octal digit or *oit*; three bits. Used as an information unit in computer engineering.
- d $= [\log 10]$. Could be called the decimal digit or *dit*. In various contexts, this unit is also known as *Bel*, *power of ten*, *order of magnitude*, *Richter-scale point*, or *decade*.

- $\mathrm{h} = [\log 16] = 4\,\mathrm{b}$. In computer engineering, the hexadecimal digit is a unit of information, which could be called a *hit*. In practice, it is called a *nibble* or *nybble*.

- $\mathrm{B} = [\log 256] = 8\,\mathrm{b} = 2\,\mathrm{h}$. The *byte* or *octet*.

- $\mathrm{kcal/mol/K} \approx 503.6\,k_\mathrm{B}/\mathrm{molecule} \approx [\log(4.9 \times 10^{218})]/\mathrm{molecule}$. In chemistry, the *kilocalorie per mole per degree Kelvin* is a common unit of (per-molecule) thermodynamic entropy, equivalent to about $503.6\,k_\mathrm{B}$ (or nats or Nepers) per molecule.

- $\mathrm{kb} = 1{,}000\,\mathrm{b} = [\log 2^{1000}]$. An information unit known as a *kilobit* in telecommunications.

- $\mathrm{kb} = 1{,}024\,\mathrm{b} = [\log 2^{2^{10}}]$. An information unit called a *kibibit*, also known as a *kilobit* in computer engineering.

- Multiplying the above definitions by 8 gives the standard definitions for the *kilobyte* unit of information, in the telecommunication and computer-engineering contexts respectively.

- Similarly for higher powers of 1,000 (or 1,024), with prefixes mega- (M), giga- (G), tera- (T), peta- (P), exa- (E), zetta- (Z), and yotta- (Y).

- $\mathrm{J/K} \approx 7.243 \times 10^{22} k_\mathrm{B} \approx 11\,\mathrm{ZB} \approx [\log 10^{3.14558 \times 10^{22}}]$. In thermodynamics, the *Joule per Kelvin* is a common unit of bulk thermodynamic entropy. Converted into information units, it is about 11 zettabytes, meaning $11 \times 1{,}024^7\,\mathrm{B}$.

- $\mathrm{kcal/K} = 4186.8\,\mathrm{J/K} \approx 3.03 \times 10^{26} k_\mathrm{B} \approx 45.2\,\mathrm{YB}$. In chemistry, the *kilocalorie per degree Kelvin* is a common unit of bulk thermodynamic entropy. In information units, it is about 45 yottabytes, meaning $45 \times 1{,}024^8\,\mathrm{B}$.

Of course, one could easily define and name still larger (or smaller) logarithmic units by applying larger (or smaller) order-of-magnitude prefixes.

The point of this exercise is to emphasize that all of the supposed disparate logarithmic scales that are in use in these various fields are ultimately all just different views of the *same* fundamental logarithmic scale. The various quantities and units discussed on all of these logarithmic scales are all exactly comparable with each other.

Among the above, perhaps the quantity whose status as a logarithmic quantity is least widely appreciated is the quantity known as *thermodynamic entropy*. Reviewing why this "physical" quantity is indeed at root truly just a logarithmic quantity is the subject of the next section.

6 Logarithmic Units and Entropy

The original definition of the quantity known as *entropy*, first introduced by Rudolph Clausius in the mid-1800s, was (in differential form)

$$\mathrm{d}S = \mathrm{d}Q/T \tag{2}$$

where $\mathrm{d}Q$ represents an infinitesimal increment of heat energy added to or removed from a system, and T is the temperature of the system. Although this is only a differential definition, we can presume that a physical system has a property called its total entropy S, changes of which correspond to the increments $\mathrm{d}S$. (However, the original definition did not specify the base value of S for any particular cases.)

Clausius observed that in any thermodynamic process, the total entropy (as he defined it) never decreased, since heat always moved spontaneously from higher-temperature systems to lower-temperature ones, and never vice-versa. He postulated that the principle of the non-decrease of entropy could be introduced as a fundamental law of physics, equivalent to the various other statements of the second law of thermodynamics (impossibility of perpetual motion machines, *etc.*).

Now, *prima facie*, Clausius' entropy does not seem in any way to be a logarithmic quantity. But with the subsequent introduction of statistical mechanics by Maxwell, Boltzmann, and Gibbs in the late 1800s, the thermodynamic entropy came to be understood as really being a statistical quantity that is naturally defined on a logarithmic scale. The Boltzmann form of the definition of entropy was expressed as

$$S = k_{\mathrm{B}} \ln W, \tag{3}$$

where W denoted the number of possible distinct microscopic ways of arranging the system (consistently with its macroscopic description), and k_{B} denoted a fundamental entropy unit which came to be called *Boltzmann's constant*, which had a value that (in conventional units of heat over temperature) was found to be equal to about $1.38 \times 10^{-23}\,\mathrm{J/K}$.

Now, the traditional stance as to the status of this equation, which is maintained today by many of the more traditional-minded thermodynamicists, is that the entropy S is fundamentally a "physical" quantity, namely a ratio of heat to temperature, and Boltzmann's equation (3) predicts what this quantity will be as a multiple of the Boltzmann's constant unit, where the multiplier is the pure number obtained from $\ln W$.

However, what is arguably the preferred (simpler and more modern) perspective on Boltzmann's equation is that it is merely a way of rendering (in traditional units) the more elegant and fundamental relation

$$S = [\log W], \tag{4}$$

where now entropy is taken as being at root just an indefinite logarithmic quantity, in the abstract sense that we outlined in the previous sections.

The modern form (4) can be seen as being exactly equivalent to (3) if we simply declare that

$$k_{\mathrm{B}} = [\log e], \tag{5}$$

since $[\log e] \ln W = [\log W]$. Note, in particular, that the choice of using $[\log e]$ as the unit in the original equation (3) was a completely arbitrary one, and was merely a consequence of the choice of using the base-e ("natural") logarithm in the formula. So, we could equally validly re-render eq. (3) in any of the following ways:

$$S = k_{\mathrm{b}} \log_2 W \tag{6}$$
$$S = k_{\mathrm{o}} \log_8 W \tag{7}$$
$$S = k_{\mathrm{d}} \log_{10} W, \tag{8}$$

where $k_{\mathrm{b}} = [\log 2] = k_{\mathrm{B}} \ln 2$, $k_{\mathrm{o}} = [\log 8] = k_{\mathrm{B}} \ln 8$, and $k_{\mathrm{d}} = [\log 10] = k_{\mathrm{B}} \ln 10$ are respectively binary, octal, and decimal entropy units, corresponding to the bases of the logarithms used. Of course, any of the other logarithmic units listed in the previous section (and a continuum of other units as well) could also have been used in Boltzmann's relation, with a suitable choice of logarithm base.

Observe now that the relations $k_{\mathrm{B}} = [\log e]$ and $k_{\mathrm{B}} \approx 1.38 \times 10^{-23}\,\mathrm{J/K}$ imply that $1\,\mathrm{K} \approx 1.38 \times 10^{-23}\,\mathrm{J}/[\log e]$, in other words, the Kelvin (or any temperature unit) is fundamentally just an expression of an amount of energy per logarithmic unit of some arbitrary size. Here it is expressed as energy per nat or Neper, where this unit quantifies the increase in the indefinite logarithm of the number of states when the state count is multiplied by e. Of course, we could equally well express the Kelvin in terms of logarithmic units of other sizes as well, for example, multiplying top and bottom by $(\ln 10)$ gives $1\,\mathrm{K} \approx 3.18 \times 10^{-23}\,\mathrm{J}/[\log 10]$, where we see we have now expressed the Kelvin in units of Joules per decade (Bel, order of magnitude, power of ten *etc.*) of increase in the number of states.

Indeed, the modern thermodynamic definition of temperature is indeed just

$$T = \mathrm{d}Q/\mathrm{d}S, \tag{9}$$

where $\mathrm{d}Q$ is the amount of heat that must be added to a system in order to increase its entropy $S = [\log W]$ by a small amount $\mathrm{d}S$, or in other words to increase its number of states by the multiplicative factor $[\exp \mathrm{d}S]$, where note we are using our indefinite exponential notation from earlier.

7 Logarithmic Units and Information

Just as with entropy, the amount of information content or information capacity I of a system can be expressed very elegantly and generically as an indefinite logarithmic quantity, that is, as

$$I = [\log W] \tag{10}$$

where W is again the number of ways of arranging the system, *e.g.*, for purposes of storing or communicating a message.

The only real distinction between entropy and information is a distinction of epistemological status.

Entropy is usually taken to refer to that part of the physical information that is *unknown*, or in other words is not included in the available overall description of a physical situation; this information is not considered part of the so-called "macrostate" of the system.

The word "information," on the other hand, is sometimes reserved to implicitly connote information that is explicitly known, although this more restricted usage is becoming less common. More and more, professionals who use the word "information" understand that physical information, in general, could have the status of being either known information, or unknown information (entropy). One word that has been proposed to indicate known information as opposed to entropy, but which is not yet very popular, is *extropy*, which was coined to serve as a complement to the word "entropy."

Of course, due to the (historically dominant) use of binary codes in our modern digital systems, information has been traditionally measured in $[\log 2]$ units (bits), or multiples thereof (such as bytes), rather than in $[\log e]$ units (nats) or $[\log 10]$ units (decades). However, as we have been emphasizing, this difference in the conventional choice of units does not at all imply that information is fundamentally a different kind of quantity from the logarithmic quantities that are used in other contexts.

In fact, we argue that the only distinction between information/entropy and other types of logarithmic quantities is that information is a logarithmic quantity derived from an absolute pure number that represents a "number of alternatives" in some sense, while most other logarithmic quantities (*e.g.* octaves, decibels, Richter scale points) are derived from pure numbers representing *ratios* between physical quantities (pitch, signal power, Earthquake strength). But fundamentally, although the sources of the pure numbers x in the two kinds of cases are different, this does not matter; the values of $[\log x]$ are always still fundamentally the very *same* type of mathematical object.

Thus, a bit of information *is*, mathematically, the very *same* kind of object as an octave of pitch. A Boltzmann's constant unit of entropy is the same kind of object as a Naper of current ratio. A decimal-digit-sized quantity of information is the *same* as a Richter-scale point of relative earthquake strength. Fundamentally, the only import of the different names for these mathematical objects is to connote their use in describing different types of situations.

Just as the number 2 is still a 2 whether we are talking about two giraffes or two potatoes, likewise the indefinite-logarithm object $[\log 2]$ is still a $[\log 2]$ unit whether we are talking about a $[\log 2]$ amount of information (called a bit) or a $[\log 2]$ size of musical interval (called an octave). And a $[\log e]$ unit is still a $[\log e]$ unit, whether we are talking about a $[\log e]$ unit of thermodynamic entropy (called Boltzmann's constant) or a $[\log e]$ unit of voltage ratios (called a Naper). A $[\log 10]$ unit is still a $[\log 10]$ unit, whether we are talking about a $[\log 10]$ unit of signal power ratio (called a Bel) or a $[\log 10]$ unit of information (called a decimal digit).

In other words, logarithmic units are logarithmic units, and logarithmic quantities (expressed by a real number times a logarithmic unit) are mathematically always the very same kind of entity, no matter the domain. The only differences are in the size of the standard units that are conventionally used in a given context, the names that we call them, and how we apply them.

8 Discussion

Today, thanks to Boltzmann and his followers, we know that a certain quantity that used to be thought of as "physical," namely entropy, is really just a "mathematical" quantity, namely an indefinite logarithmic quantity derived from the number of states, or in other words a kind of information. Since indefinite logarithmic units are scale-free, there is no natural unit or "atom" of entropy or information that we must use, only units that we choose rather arbitrarily, by convention or for mathematical or technological convenience, such as the nat (Boltzmann's constant) or the bit.

In the future, it is possible that we might discover that other quantities that are currently thought of as "physical" could at root turn out to really be logarithmic quantities as well. For example, Tommaso Toffoli has speculated that, just as entropy turned out to be equivalent to the logarithm of the number of possible states that a system could (statically) be in, perhaps *energy* could be shown in some way to really be equivalent a logarithm of the number of possible *computations* that a system could carry out dynamically at the microscale. As of this writing, this intriguing idea is still rather far from being substantiated, but the history of how our understanding of the quantity of entropy has evolved indeed makes Toffoli's proposal seem like an idea worth exploring.

Besides entropy and (perhaps) energy, one wonders whether other kinds of physical quantities such as distances and times might potentially be shown to ultimately be logarithmic quantities as well. That this might be true is hinted at by the Bekenstein formula for the entropy of a black hole, $S = A/4$, where A is the hole's event horizon area in Planck units, and S is the entropy in nats. Thus, for example, we could assign the Planck unit of length to be the square root of a nat, $\ell_{\mathrm{P}} = [\log e]^{1/2}$, and then write $A = [\log W^4]$, where W is the number of states of the black hole, and this would be consistent with the Bekenstein relation as well as the entropy relation $S = [\log W]$. However, in this line of thought it remains obscure why the area should be the indefinite logarithm of the *fourth* power of the number of states, and why the length unit should have dimensions of a square root of a logarithmic unit. Still, this may be an interesting line to pursue further.

9 Conclusion

In this paper, we have reviewed a well-defined mathematical concept of an *indefinite* logarithm function in which no particular logarithm base is selected, and have shown that the entities returned by this function can be used as the basis for a system of mathematical quantities that is exactly analogous in its behavior to systems of dimensioned physical quantities. In fact, this mathematical system of indefinite logarithmic quantities *exactly* corresponds the physical quantity known as entropy, when the logarithms are applied to the number of distinguishable physical states that are consistent with a given abstract description of the system. This quantity (the indefinite logarithm of the number of states) is also called "information" in a slightly broader context.

In other words, physical (thermodynamic) entropy really *is* nothing but (unknown) information in the physical state, and its quantity really *is* nothing other than the indefinite logarithm of the state count. Further, Boltzmann's constant k_{B} is really nothing other than a representation (in conventional physical units of energy over temperature) of the specific (and arbitrarily chosen) abstract indefinite-logarithm unit $[\log e]$, which is known as the nat or the Naper in other contexts. And, thermodynamic temperature really is nothing but the energy per logarithmic unit, for small increments in the indefinite logarithm of the state count.

There are speculations that other quantities such as energy that we presently think of as being fundamentally "physical" in nature (as opposed to mathematical) might (similarly to entropy) someday be revealed to be, at root, derived from logarithmic quantities of some sort.

Of course, if physics can someday be *exactly* described by mathematics, as most theoretical physicists believe (or at least hope), then ultimately, the entire distinction between mathematical and physical quantities becomes somewhat of an artificial and illusory one, since we cannot then rule out the possibility that our physical universe may really *be* nothing but a particular (very elaborate) mathematical structure, one in which we (and our thought processes) happen to be embedded. What is a "physical" quantity then ultimately becomes only a question of which mathematical quantities happen to arise naturally within the context of the particular mathematical structures that make up our physical universe.

To conclude, although there are probably no substantive ideas in this paper that have not been said many times before, somewhere in the literature (though perhaps in different terms), and although many of these ideas would likely be considered self-evident to professional mathematicians, we nevertheless felt that many of these ideas lack exposure at present within certain communities, and that it would be worthwhile to present and explain them again, so as to facilitate the more widespread understanding of these issues. We hope that this paper serves that purpose, at least.

10 References

The reference list is still under construction. The author would appreciate receiving from readers suggested references to appropriate prior sources that discuss these or similar ideas, so that he can cite the sources in future versions of this paper, as well as in future papers on related topics.