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Lecture #9: AC Network Analysis (Part III)

EEL 3003
Introduction to Electrical Engineering
Summer Semseter, 2013

Instructor: Dr. Michael Frank

Administrative Announcements

- Outline of today's class session:
  1. Finish Chapter 4, AC Network Analysis
     - §4.4 – Phasor Solution of Circuits with Sinusoidal Excitation
     - §4.5 – AC Circuit Analysis Methods
- Homework reminder:
  - Read Ch. 4 & practice w. these exercises:
    - 4.1*, 4.12*, 4.31, 4.59*, 4.66, 4.68, 4.71, 4.77*
  - Quiz will be this Thursday (June 13th).

§4.4 – Phasor Solution of Circuits with Sinusoidal Excitation

- An abstract method for solving dynamic, linear circuits
  - Made of ideal resistors, capacitors, & inductors

phas*or* ≠ Star Trek phas*er*

- *Given* sinusoidal excitation of the circuit by current/voltage sources at a single frequency
  - But, this method extends to other periodic sources using Fourier decomposition & superposition
    - We may or may not have time to get that deep into it…

Review of Complex Numbers

- Why complex numbers?
  - Technical answer (in abstract algebra lingo):
    - The complex number system is *the unique* associative, commutative division algebra that is closed under addition, multiplication, and exponentiation.
      - *I.e.*, it's not just some arbitrary made-up system; it's fundamental!
        - No other set of rules can "complete" the system of real arithmetic

cf. Frobenius theorem

$$\mathrm{j} \equiv \sqrt{-1}$$
(imaginary unit)

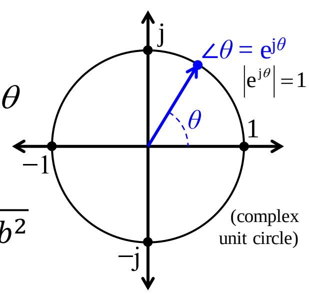
$$z = a + \mathrm{j}b$$
(complex number)

$$\mathrm{Re}[a+\mathrm{j}b] = a$$
$$\mathrm{Im}[a+\mathrm{j}b] = b$$
(real and imaginary parts)

$$\angle\theta = \mathrm{e}^{\mathrm{j}\theta} = \cos\theta + \mathrm{j}\cdot\sin\theta$$
(Euler's formula)

$$(a+\mathrm{j}b)^* = a - \mathrm{j}b$$
(definition of complex conjugate)

$$|a + \mathrm{j}b| = \sqrt{a^2 + b^2}$$
(definition of norm)



(complex plane)

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Conversion Between Rectangular & Polar Form of Complex Numbers

- Rectangular form (real & imaginary parts):
  - $z = (a + jb)$
    - $a = \mathrm{Re}[z] = m \cos\theta$ - The real part of z.
    - $b = \mathrm{Im}[z] = m \sin\theta$ - The imaginary part of z.
- Polar form (phase-magnitude representation):
  - $z = m \cdot e^{i\theta}$
    - $m = |z| = \sqrt{a^2 + b^2}$
      - Magnitude, modulus, norm, amplitude, absolute value of z
    - $\theta = \arg z = \mathrm{atan2}(a, b) = 2\tan^{-1}\frac{b}{m+a}$ When $a > 0$, $\theta = \tan^{-1}(b/a)$
      - The argument, angle, phase of z.

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Complex Addition & Multiplication

- Given complex numbers:
  - $z_1 = a_1 + jb_1 = m_1 e^{j\theta_1} = m_1\angle\theta_1$,
  - $z_2 = a_2 + jb_2 = m_2 e^{j\theta_2} = m_2\angle\theta_2$,
- Addition and multiplication of them works like so:
  - Addition is like vector addition:
    - $z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2)$
  - To multiply, multiply magnitudes & add angles:
    - $z_1 z_2 = m_1 e^{j\theta_1} m_2 e^{j\theta_2} = (m_1 m_2) e^{j(\theta_1+\theta_2)} = m_1 m_2 \angle(\theta_1 + \theta_2)$
    - Or in terms of components:
      - $z_1 z_2 = (a_1 + jb_1)(a_2 + jb_2) = (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + b_1 a_2)$

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Complex Division

- Given a complex number
  - $z = a + \mathrm{j}b = m\mathrm{e}^{\mathrm{j}\theta} = m\angle\theta,$
- Its reciprocal (multiplicative inverse) is:
  - $z^{-1} = \frac{1}{m}\mathrm{e}^{-\mathrm{j}\theta} = \frac{z^*}{m^2} = \frac{a-\mathrm{j}b}{a^2+b^2}$
- To divide two complex numbers z_1/z_2,
  - Divide their magnitudes and subtract their angles:
    - $\frac{z_1}{z_2} = \frac{m_1\mathrm{e}^{\mathrm{j}\theta_1}}{m_2\mathrm{e}^{\mathrm{j}\theta_2}} = \frac{m_1}{m_2}\mathrm{e}^{\mathrm{j}(\theta_1-\theta_2)} = (m_1/m_2)\angle(\theta_1-\theta_2)$
    - Or in terms of components:
      - $\frac{z_1}{z_2} = \frac{z_1 {z_2}^*}{{m_2}^2} = \frac{(a_1+\mathrm{j}b_1)(a_2-\mathrm{j}b_2)}{{a_2}^2+{b_2}^2} = \frac{(a_1a_2+b_1b_2)-\mathrm{j}(a_1b_2-b_1a_2)}{{a_2}^2+{b_2}^2}$

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Phasors

- Suppose you have a general sinusoidal signal (current or voltage):

$$x(t) = A\cos(\omega t + \theta)$$

- Then $x(t)$ is fully determined by its frequency ω and the complex number $\mathbf{X} = A\angle\theta = A\mathrm{e}^{\mathrm{j}\theta}$.

$$\begin{aligned} x(t) &= A\cos(\omega t+\theta) = \mathrm{Re}[A\cos(\omega t+\theta) + \mathrm{j}A\sin(\omega t+\theta)] \\ &= \mathrm{Re}[A\mathrm{e}^{\mathrm{j}(\omega t+\theta)}] = \mathrm{Re}[A\mathrm{e}^{\mathrm{j}\omega t+\mathrm{j}\theta}] = \mathrm{Re}[A\mathrm{e}^{\mathrm{j}\omega t}\mathrm{e}^{\mathrm{j}\theta}] \\ &= \mathrm{Re}[(A\mathrm{e}^{\mathrm{j}\theta})\mathrm{e}^{\mathrm{j}\omega t}] = \mathrm{Re}[\mathbf{X}\mathrm{e}^{\mathrm{j}\omega t}] \end{aligned}$$

X is sometimes written as $\mathbf{X}(\mathrm{j}\omega)$

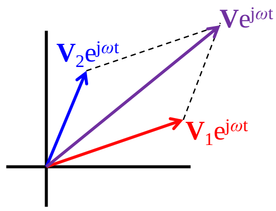
Adding Sinusoids Using Phasors

- Given two sinusoidal voltage sources $v_1(t)=V_1\cos(\omega t+\theta_1)$ and $v_2(t)=V_2\cos(\omega t+\theta_2)$ at the same frequency ω, find their sum $v(t) = v_1+v_2$.
- Solution: Just add their phasors!
  - This works because:

$$\mathbf{V}_1 = V_1\angle\theta_1,\ \mathbf{V}_2 = V_2\angle\theta_2$$

$$\mathbf{V} = V\angle\theta = \mathbf{V}_1+\mathbf{V}_2$$

$$\begin{aligned} v(t) &= v_1(t)+v_2(t) \\ &= \mathrm{Re}[\mathbf{V}_1 e^{j\omega t} + \mathbf{V}_2 e^{j\omega t}] \\ &= \mathrm{Re}[(\mathbf{V}_1 + \mathbf{V}_2) e^{j\omega t}] \\ &= \mathrm{Re}[\mathbf{V} e^{j\omega t}] \end{aligned}$$



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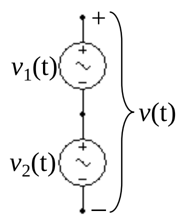
Adding Sinusoids Example

- Suppose we have the following two AC voltage sources in series:
  - $v_1(t) = (2\text{ V})\cos(\omega t + \pi/3)$
  - $v_2(t) = (3\text{ V})\cos(\omega t + \pi/2)$
- What is their sum,
  - $v(t) = v_1(t) + v_2(t)$,
- expressed as a simple sinusoidal function?
- *Do this as an in-class exercise…*



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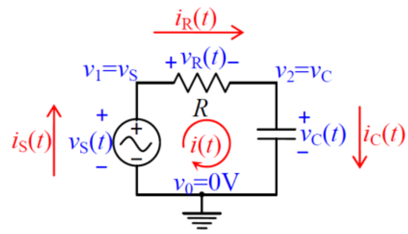
Practice Exercise #1

- Do this at home…
  - In the previous lecture,
    - see also supplemental "Notes on Solving AC Circuits"
  - we solved the differential equations for this circuit and got this equation for $v_C(t)$:



$$v_C(t) = \frac{V}{1+(\omega RC)^2}\sin \omega t - \frac{V\omega RC}{1+(\omega RC)^2}\cos \omega t$$

- Now, if $V = 9V$, $f = 100$ Hz, $R = 1$ kΩ, and $C = 1$ μF,
  1. Use phasors to express $v_C(t)$ using a *single* sinusoidal function.
  2. By how many degrees does $v_C(t)$ lag behind $v_S(t)$?

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Superposition of AC Signals

- What if not all of the AC sources in a circuit are at the same frequency?
  - We can still solve the circuit, using the superposition principle, as follows:
    - Solve the circuit separately for each source
      - Zeroing out the other sources
    - Add together the solutions

Impedance

- A complex-valued extension of the concept of resistance, for AC circuits: **Z** = **V**/**I** (phasors)
  - Captures how capacitors & inductors can impede (in the sense of delay) the phase of AC signals
    - in addition to how resistors directly impede (in the sense of resist) the flow of current
- Impedance **Z** = *R* + j*X* breaks down as:
  - Real part *R* = Re[**Z**] = Resistance (always +).
  - Imag. part *I* = Im[**Z**] = Reactance (+ or −).

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Impedance of a Resistor

- WLOG, assume voltage phasor of signal source is:
  - $\mathbf{V}_S = V\angle 0$ (set phase of source = 0)
- For a resistor:
  - $\mathbf{V} = V\angle 0$, $\mathbf{I} = \frac{V}{R}\angle 0$, ∴

$$\mathbf{Z} = \frac{V\angle 0}{\left(V/_R\right)\angle 0} = R\angle 0 = R \text{ (real)}$$

Impedance of an Inductor

- Recall diff. eq. of inductor: $v(t) = L\frac{di(t)}{dt}$.
- For $\mathbf{V}=V\angle 0$, $v(t) = V\cos\omega t$.
  - Thus, $i(t) = \frac{V}{\omega L}\sin\omega t = \frac{V}{\omega L}\cos(\omega t - {}^{\pi}/_{2})$,
  - so $\mathbf{I} = \frac{V}{\omega L}\angle -\frac{\pi}{2}$.
- Thus, an inductor's impedance is:

$$\mathbf{Z} = \frac{V\angle 0}{({}^{V}/_{\omega L})\angle -{}^{\pi}/_{2}} = \omega L\angle\frac{\pi}{2} = \mathrm{j}\omega L \text{ (imag.)}$$

  - Thus, the reactance of an inductor is $X = \omega L$.

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Impedance of a Capacitor

- Recall diff. eq. of capacitor: $i(t) = C\frac{dv(t)}{dt}$.
- For $\mathbf{V}=V\angle 0$, $v(t) = V\cos\omega t$.
  - Thus, $i(t) = -\omega CV\sin\omega t = \omega CV\cos(\omega t + {}^{\pi}/_{2})$,
  - so $\mathbf{I} = \omega CV\angle{}^{\pi}/_{2}$.
- Thus, a capacitor's impedance is:

$$\mathbf{Z} = \frac{V\angle 0}{\omega CV\angle{}^{\pi}/_{2}} = \frac{1}{\omega C}\angle -\frac{\pi}{2} = -\mathrm{j}\frac{1}{\omega C} = \frac{1}{\mathrm{j}\omega C}.$$

  - Thus, the reactance of a capacitor is $X = -{}^{1}/_{\omega C}$.

Admittance

- The reciprocal of impedance, $\mathbf{Y} = 1/\mathbf{Z}$.
  - For a capacitor: $\mathbf{Y} = \mathrm{j}\omega C$.
  - For an inductor: $\mathbf{Y} = {}^{1}/_{\mathrm{j}\omega L}$.
- Like with impedance, we can break down admittance into real & imaginary parts:

$$\mathbf{Y} = G + \mathrm{j}B.$$

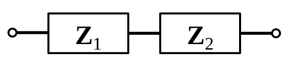
$$G = \frac{R}{|Z|^2} = \frac{R}{R^2 + X^2}$$

$$B = \frac{-X}{|Z|^2} = \frac{-X}{R^2 + X^2}$$

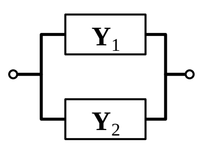
  - G = (AC version of) conductance
  - B = susceptance
- If X=0, G=1/R; if R=0, B=−1/X.

Combining Elements with Impedance

- For elements in series, their impedances add:
  - $\mathbf{Z} = \mathbf{Z}_1 + \mathbf{Z}_2$



- For elements in parallel, their admittances add:
  - $\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2$
  - Thus $\mathbf{Z} = \frac{1}{\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}}$



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§4.5 – AC Circuit Analysis Methods

- Here's a general procedure for analyzing all linear circuits excited by AC sources:
  1. Separate sources by frequency (treat separately)
  2. Convert source functions $v(t)/i(t)$ to phasors **V**/**I**
  3. Convert all elements to complex impedances.
  4. Solve with usual methods like in ch. 3
     - Simplification of subcircuits via equivalent circuits
     - Node voltage or mesh current analysis → linear eqs.
  5. Convert results from phasor to time-domain form
  6. Superpose solutions for different frequencies

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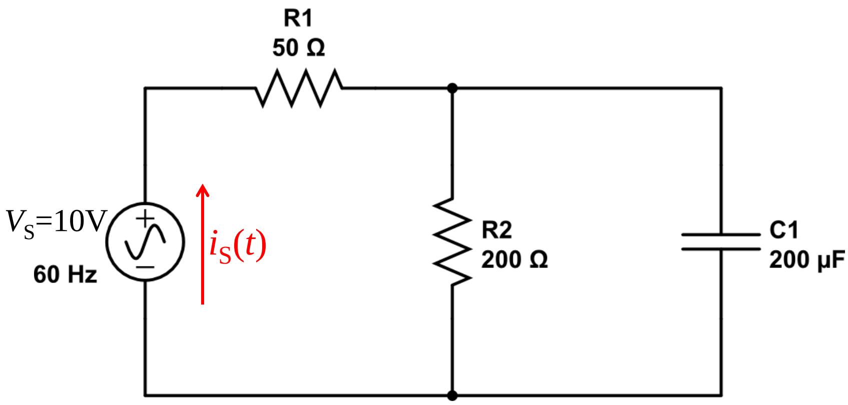
AC Analysis Example

- Problem: Find the source current $i_S(t)$.



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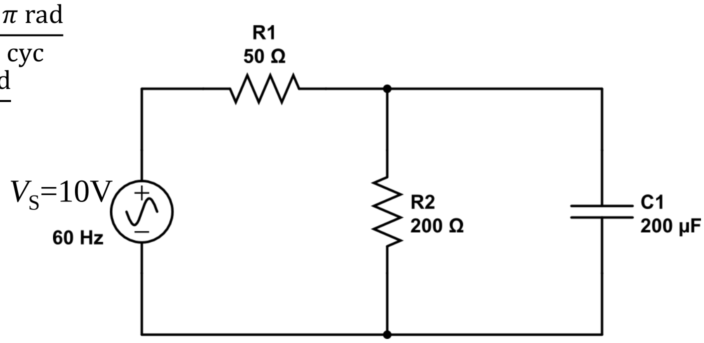
1. Identify source's voltage function & its corresponding phasor

- Assume source phase is 0 (not otherwise given)
  - $v_S(t) = 10 \cos \omega t$; $\mathbf{V} = 10\angle 0$.

$$\omega = 60\frac{\text{cyc}}{\text{s}} \cdot \frac{2\pi\ \text{rad}}{\text{cyc}} = 377\frac{\text{rad}}{\text{s}}$$



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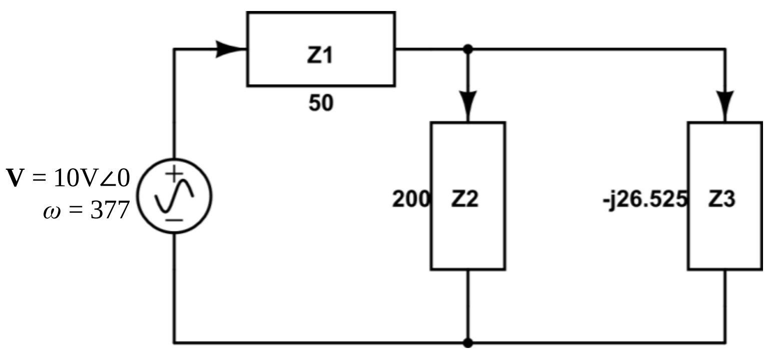
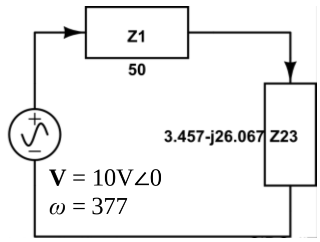
2. Express each element as a complex impedance

$$\mathbf{Z}_3 = \frac{1}{\text{j}\omega C} = \frac{-\text{j}}{(377\ \text{s}^{-1})(100\ \mu\text{F})} = -\text{j} \cdot 26.526\ \Omega$$



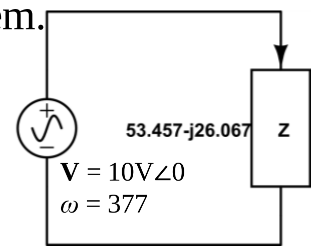
3. Simplify circuit by combining elements

- $\mathbf{Z}_2$ and $\mathbf{Z}_3$ are in parallel, so we can combine them by adding their admittances:
  - $\mathbf{Y}_2 = 1/\mathbf{Z}_2 = 1/200\Omega = 0.005$ S.
  - $\mathbf{Y}_3 = 1/\mathbf{Z}_3 = \mathrm{j}\cdot 0.0377$ S
  - $\mathbf{Y}_2 + \mathbf{Y}_3 = (0.005 + \mathrm{j}\cdot 0.0377)$S
  - $\mathbf{Z}_2 \| \mathbf{Z}_3 = 1/(\mathbf{Y}_2 + \mathbf{Y}_3) = (3.457 - \mathrm{j}\cdot 26.067)\ \Omega$.



Circuit simplification continued…

- $\mathbf{Z}_1$ and $\mathbf{Z}_{23}$ are in series, so we can combine their impedances by adding them.
  - For the overall load:
    - $\mathbf{Z} = \mathbf{Z}_1 + \mathbf{Z}_{23}$
      $= (50\ \Omega) + (3.457 - \mathrm{j}\cdot 26.067)\ \Omega$
      $= (53.457 - \mathrm{j}\cdot 26.067)\ \Omega$



- Now we can apply $\mathbf{I} = \mathbf{V}/\mathbf{Z}$:
  - $\mathbf{I} = (10\mathrm{V}\angle 0)/[(53.457 - \mathrm{j}\cdot 26.067)\ \Omega$
    $= (0.151 + \mathrm{j}\cdot 0.0737)\ \mathrm{A} = 0.168\angle 0.454\ \mathrm{A}$

  0.454 rad = 26°
- Thus, $i_S(t) = (0.168\ \mathrm{A})\cos(377\ t/\mathrm{s} + 0.454)$. □

Thévenin/Norton Equivalents of AC Subcircuits

- Again, this is just like the DC case,
  - but with complex impedances in place of resistances.
    - Thévenin equivalent circuit:
      - AC voltage source in series with a **Z** element.
    - Norton equivalent circuit:
      - AC current source in parallel with a **Z** element.