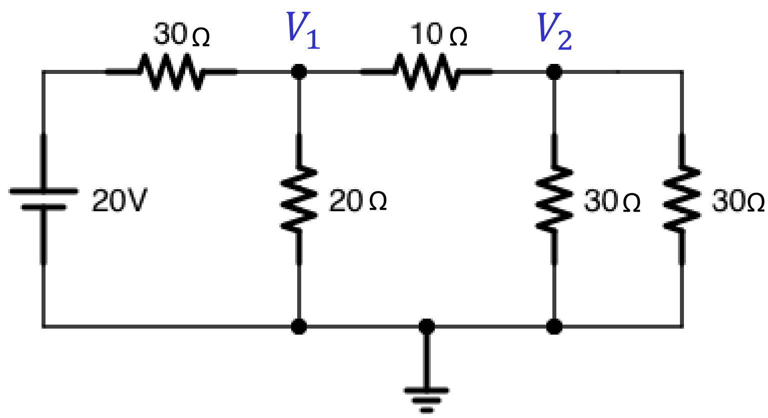
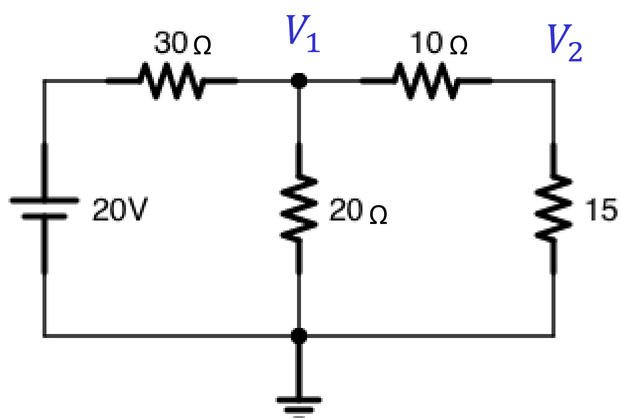
Node Voltage & Mesh Current Analysis

Notes on Textbook Sections 3.2-3.4.

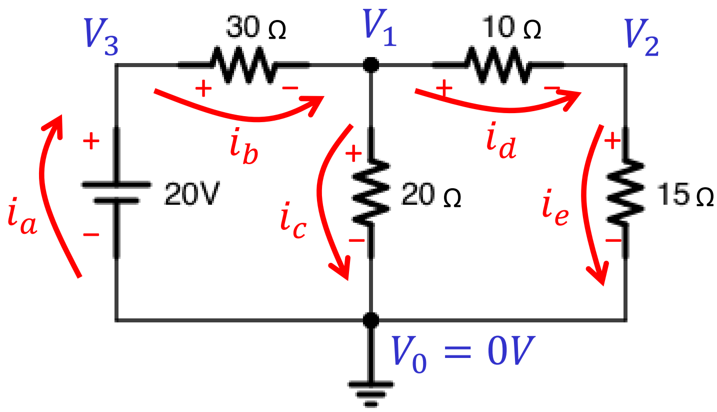
Example #3. Let's consider problem 3.2, which says to use node voltage analysis to find the voltages V_1 and V_2 in this circuit:



First, we note that we can simplify the circuit a bit by combining the two parallel 30 Ω resistors on the right into a single 15 Ω resistor:



Next, we need to label the various branch currents, and the other unlabeled node voltages:



Next, we write down the KCL equations for nodes #1, 2, & 3:

(1) $i_b - i_c - i_d = 0.$
(2) $i_d - i_e = 0.$
(3) $i_a - i_b = 0.$

Next, we transform these to equations in terms of node voltages using Ohm's Law, except in the case of the source current i_a, which is an unknown current which will be found by our analyis.

(1) $\frac{V_3 - V_1}{30\Omega} - \frac{V_1}{20\Omega} - \frac{V_1 - V_2}{10\Omega} = 0.$
(2) $\frac{V_1 - V_2}{10\Omega} - \frac{V_2}{15\Omega} = 0.$
(3) $i_a - \frac{V_3 - V_1}{30\Omega} = 0.$

Next, since the voltage $V_3 = V_0 + 20V = 0V + 20V = 20$ V is known due to the constant-voltage source, we can substitute a constant 20V for it everywhere it appears:

(1) $\frac{20\text{V} - V_1}{30\Omega} - \frac{V_1}{20\Omega} - \frac{V_1 - V_2}{10\Omega} = 0.$
(2) $\frac{V_1 - V_2}{10\Omega} - \frac{V_2}{15\Omega} = 0.$
(3) $i_a - \frac{20\text{V} - V_1}{30\Omega} = 0.$

Next, we rearrange the equations to distribute the denominators over their respective numerator expressions, and gather together terms involving each variable:

(1) $\frac{20\text{V}}{30\Omega} - \frac{V_1}{30\Omega} - \frac{V_1}{20\Omega} - \frac{V_1}{10\Omega} - \frac{V_2}{10\Omega} = 0.$
(2) $\frac{V_1}{10\Omega} - \frac{V_2}{10\Omega} - \frac{V_2}{15\Omega} = 0.$
(3) $i_a - \frac{20\text{V}}{30\Omega} + \frac{V_1}{30\Omega} = 0.$

Next, we factor out the variables and group together the coefficients, and put the constant terms on the right-hand side of the equations, and make sure there is a term for each variable i_a, V_1, V_2 in each equation:

$$(1)\ \ (0)i_a + \left(-\frac{1}{30\Omega} - \frac{1}{20\Omega} - \frac{1}{10\Omega}\right)V_1 - \left(\frac{1}{10\Omega}\right)V_2 = -\tfrac{2}{3}\text{A}.$$
$$(2)\ \ (0)i_a + \left(\frac{1}{10\Omega}\right)V_1 + \left(-\frac{1}{10\Omega} - \frac{1}{15\Omega}\right)V_2 = 0.$$
$$(3)\ \ (1)i_a + \left(\frac{1}{30\Omega}\right)V_1 + (0)V_2 = \tfrac{2}{3}\text{A}.$$

Transform the sums of fractions using least-common-multiples of the denominators:

$$(1)\ \ (0)i_a + \left(-\frac{2}{60\Omega} - \frac{3}{60\Omega} - \frac{6}{60\Omega}\right)V_1 - \left(\frac{1}{10\Omega}\right)V_2 = -\tfrac{2}{3}\text{A}.$$
$$(2)\ \ (0)i_a + \left(\frac{1}{10\Omega}\right)V_1 + \left(-\frac{3}{30\Omega} - \frac{2}{30\Omega}\right)V_2 = 0.$$
$$(3)\ \ (1)i_a + \left(\frac{1}{30\Omega}\right)V_1 + (0)V_2 = \tfrac{2}{3}\text{A}.$$

Sum up the sums of fractions and reduce them to simplest form.

$$(1)\ \ (0)i_a + \left(-\frac{11}{60\Omega}\right)V_1 - \left(\frac{1}{10\Omega}\right)V_2 = -\tfrac{2}{3}\text{A}.$$
$$(2)\ \ (0)i_a + \left(\frac{1}{10\Omega}\right)V_1 + \left(-\frac{1}{6\Omega}\right)V_2 = 0.$$
$$(3)\ \ (1)i_a + \left(\frac{1}{30\Omega}\right)V_1 + (0)V_2 = \tfrac{2}{3}\text{A}.$$

At this point, we are ready to re-express these three equations as a matrix equation. We remove the units (ohms and amps) from the constants for ease of readability. Remember, though, that they are implicitly still there (since these are equations about dimensioned physical units).

$$\begin{bmatrix} 0 & -{}^{11}\!/_{60} & -{}^{1}\!/_{10} \\ 0 & {}^{1}\!/_{10} & -{}^{1}\!/_{6} \\ 1 & {}^{1}\!/_{30} & 0 \end{bmatrix} \begin{bmatrix} i_a \\ V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -{}^{2}\!/_{3} \\ 0 \\ {}^{2}\!/_{3} \end{bmatrix}.$$

At this point, we can easily directly solve the equations in Scilab:

```
-->inv([0,-11/60,1/10;0,1/10,-
1/6;1,1/30,0])*[-2/3;0;2/3]
 ans  =

    0.4864865
    5.4054054
    3.2432432
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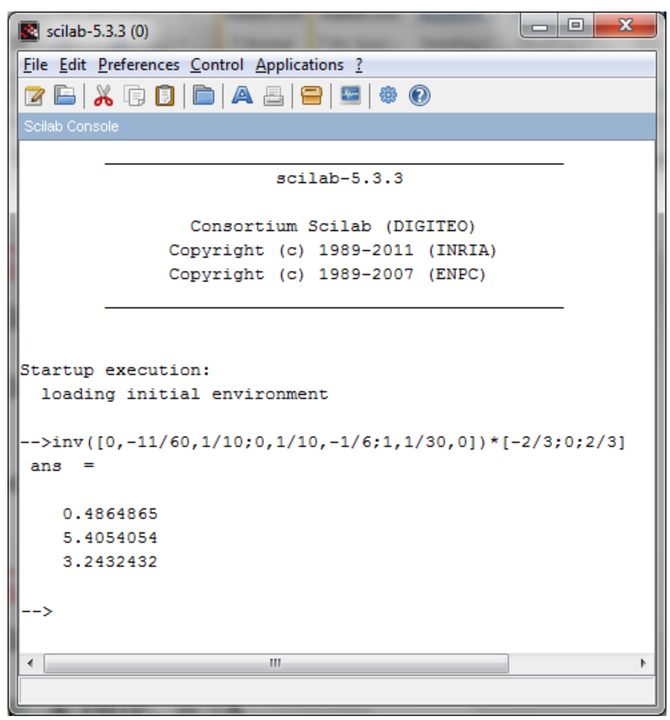

Or in other words (adding the units back in, and rounding to three decimal places), we have that:

$i_a = 0.486$ A.

$V_1 = 5.405$ V.

$V_2 = 3.243$ V.

To check this result, we can compute the branch currents and directly verify that KCL is satisfied in each case:

$i_b = (V_3 - V_1)/30\Omega = (20\text{ V} - 5.405\text{ V})/30\Omega = 0.486\text{ A} = i_a.$ ✓ (KCL node 3)

$i_c = V_1/20\Omega = 5.405\text{ V}/20\Omega = 0.270\text{ A},$

$i_d = (V_1 - V_2)/10\Omega = (5.405\text{ V} - 3.243\text{ V})/10\Omega = 0.216\text{ A},$

$i_c + i_d = 0.270\text{ A} + 0.216\text{ A} = 0.486\text{ A} = i_b.$ ✓ (KCL node 1)

$i_e = V_2/15\Omega = 3.243\text{ V}/15\Omega = 0.216\text{ A} = i_d.$ ✓ (KCL node 2)

At this point, since all of the KCL equations check out, we can be confident that we indeed solved the problem correctly.

Incidentally, these same results were also found by the iCircuit app on my iPhone, which I used to draw the circuit schematics above – this gives another way you can check your results. Here's a screenshot of that app: