Notes on AC Circuit Analysis

EEL 3003 (Intro. to Elec. Eng.), Tianjin, China, Summer B 2012

1. Energy-Storage (Dynamic) Circuit Elements (a.k.a. Reactances) [Textbook §4.1]

1.1. (Ideal) Capacitors

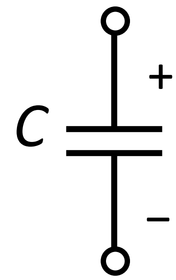
Classical capacitor structure:

Two parallel conductive plates, each with surface area *A* (on each side), separated from each other at distance *d* by air or other dielectric (insulator) with permittivity ε.

Capacitance of this structure (for small *d*, i.e., ignoring fringe capacitances) is:

$$C = \frac{\varepsilon A}{d}.$$

Capacitor <u>Symbol</u>



Standard unit of capacitance: 1 Farad (F) = 1 C/V. Also commonly used: µF, nF, pF.

Example:

Two (1 m)×(1 m) thin square metal plates, spaced 1 mm apart, with air between them (permittivity $\varepsilon = 8.854\times10^{-12}$ F/m) would have an overall capacitance (ignoring fringes) of:

$$C = \frac{(8.854\times10^{-12}\ \mathrm{F/m})(1\ \mathrm{m}^2)}{10^{-3}\ \mathrm{m}} = 8.854\times10^{-9}\ \mathrm{F} = 8.854\ \mathrm{nF}.$$

Capacitor equations. The quantity *Q* of separated charge (of each sign, + and −) stored on the plates of capacitor when the voltage across it is *V* is:

$$Q = CV.$$

The same relation also holds instantaneously in the case of time-varying voltages:

$$q(t) = C\cdot v(t).$$

Taking the derivative of that equation with respect to time yields an equation for the instantaneous current intensity *i*(*t*) through the capacitor:

$$\frac{\mathrm{d}}{\mathrm{d}\,t}q(t) = \frac{\mathrm{d}}{\mathrm{d}\,t}C\cdot v(t)$$

$$\dot{q}(t) = C\frac{\mathrm{d}}{\mathrm{d}\,t}v(t)$$

$$i(t) = C\frac{\mathrm{d}\,v(t)}{\mathrm{d}\,t}.$$

Thus, the current through a capacitor is proportional to the time *derivative* of the voltage, as opposed to (in the case of a resistor) the voltage itself.

If we let the capacitor's voltage at some initial time t_0 be V_0, then we can integrate both sides of the above equation to find its voltage at any other time t, as a functional of the time-varying current $i(t)$ through the capacitor over times between t_0 and t:

$$\int_{\tau=t_0}^{t} i(\tau)\ \mathrm{d}\,\tau = \int_{\tau=t_0}^{t} C\frac{\mathrm{d}\,v(\tau)}{\mathrm{d}\,\tau}\ \mathrm{d}\,\tau$$

$$\int_{\tau=t_0}^{t} i(\tau)\ \mathrm{d}\,\tau = C\int_{\tau=t_0}^{t} \mathrm{d}\,v(\tau)$$

$$\int_{\tau=t_0}^{t} i(\tau)\ \mathrm{d}\,\tau = C[v(t) - v(t_0)]$$

$$\frac{1}{C}\int_{\tau=t_0}^{t} i(\tau)\ \mathrm{d}\,\tau = v(t) - V_0$$

$$v(t) = V_0 + \frac{1}{C}\int_{\tau=t_0}^{t} i(\tau)\ \mathrm{d}\,\tau.$$

Series/parallel combination of capacitors. Capacitances in parallel add. The reciprocal of capacitance is called elastance (E), and its (unofficial) unit is sometimes called the "daraf" (D = 1/F). We can write the equation $V = EQ$ for a capacitor with elastance E charged up with charge Q. For capacitors in series, current is the same for all (due to KCL) and thus Q is the same for all (if initially uncharged, Q=0), and by KVL the voltages add in series, and thus the elastances of capacitors in series add, so that capacitances in series combine like resistances in parallel:

$$C = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}}.$$

Energy storage in capacitors. Using W to denote the work performed in charging up a capacitor from voltage 0 at time 0 to voltage V at time t, we can find it by integrating the instantaneous power transferred to the capacitive load:

$$W = \int_{\tau=0}^{t} P(\tau)\mathrm{d}\,\tau$$

$$= \int_{\tau=0}^{t} i(\tau)\cdot v(\tau)\,\mathrm{d}\,\tau$$

$$= \int_{\tau=0}^{t} C\frac{\mathrm{d}\,v(\tau)}{\mathrm{d}\,\tau}\cdot v(\tau)\,\mathrm{d}\,\tau$$

$$= C\int_{\tau=0}^{t} v(\tau)\,\mathrm{d}\,v(\tau)$$

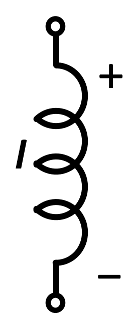
$$= C\left[\left(\frac{1}{2}[v(\tau)]^2\right)\Big|_{\tau=0}^{t}\right]$$

$$= C\left(\frac{1}{2}[v(t)]^2\right)$$

$$= \frac{1}{2}CV^2.$$

And the same amount of energy is transferred out of the capacitor (work performed by the capacitor on the environment) when discharging the capacitor from voltage V back to 0. The direction of power flow is always *into* the capacitor when its voltage is moving *away* from 0 (in either direction), and *out of* the capacitor when its voltage is moving back *towards* 0 (from either direction). No power is dissipated in an ideal capacitor when it is charged or discharged. (Real capacitors, however, always have some parasitic resistance in their leads, and thus there is non-zero dissipation when charging them at finite rates.)

Inductor Symbol:



1.2. (Ideal) Inductors

Classical inductor structure: Coil of thin metal wire embedded in an insulating medium (e.g., air). With N turns, cross-sectional area A, length ℓ, and permeability of medium μ, the inductance of the coil is:

$$L = \frac{\mu N^2 A}{\ell}.$$

Standard unit of inductance: 1 Henry (H) = 1 V·s/A.

Also commonly used: mH, μH.

Example:

Consider a wire of length $\ell = 1$ m, coiled in vacuum ($\mu = \mu_0 = 4\pi\cdot10^{-7}$ H/m). If the coil has N turns, then the circumference of the (assumed circular) coil is $c = \ell/N$, its radius is $r = \ell/2\pi N$, and so its cross-sectional area is $A = \pi r^2 = \ell^2/4\pi N^2$. So the formula for its inductance simplifies to just $L = \mu\ell/4\pi = 10^{-7}\mathrm{H} = 0.1\ \mu\mathrm{H}$.

Inductor equations:

$$v(t) = L\frac{\mathrm{d}\,i(t)}{\mathrm{d}\,t}.$$

The voltage across the inductor is proportional to the rate of change of current through it. Note the symmetry with the corresponding capacitor equation.

$$i(t) = I_0 + \frac{1}{L}\int_{\tau=t_0}^{t} v(\tau)\ \mathrm{d}\,\tau.$$

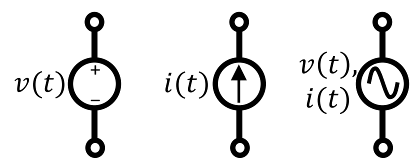
The instantaneous current $i(t)$ through an inductor can be calculated from the initial current and the time-integral of the voltage.

$$W = \frac{1}{2}LI^2.$$

Work to "boost up" an ideal inductor with inductance L from zero current to to a current intensity of I. The same amount of work will be provided by the inductor when its current falls from I back to 0.

2. Time-Dependent Sources [§4.2]

Symbols. Pictured at right are a general time-dependent voltage source, a general time-dependent current source, and a sinusoidal source, which may be defined in terms of either a time-varying voltage $v(t)$ or current $i(t)$.



Periodic signals. A (voltage or current) signal $x(t)$ that is *periodic* (not necessarily sinusoidal) obeys the following equation for all integers n and all values of t:

$$x(t) = x(t + nT)$$

where T is the repetition period (or just *period*) of the signal $x(t)$. The frequency of repetition (or just *frequency*) of the signal is then just the reciprocal of the period; it can be expressed as cycles per unit time, or simply in inverse time units:

$$f = \frac{1(\text{cyc})}{T} = \frac{1}{T}.$$

Meanwhile, the *angular frequency* or *radian frequency* $\omega(t)$ is just f times 2π, the number of radians in a cycle; it is in units of angle per unit time, although the angular units are often omitted:

$$\omega(t) = \frac{2\pi \text{ rad}}{1 \text{ cyc}} f = 2\pi f.$$

When working with angular units, keep in mind the following identity:

$$1 \text{ cyc} = 2\pi \text{ rad} = 360°.$$

Thus, the following unitary ratios of angular units can be used as convenient conversion factors:

$$\frac{1 \text{ cyc}}{2\pi \text{ rad}} = \frac{1 \text{ cyc}}{360°} = \frac{2\pi \text{ rad}}{1 \text{ cyc}} = \frac{2\pi \text{ rad}}{360°} = \frac{360°}{1 \text{ cyc}} = \frac{360°}{2\pi \text{ rad}} = 1.$$

Types of periodic signals. Some commonly-encountered periodic signals, in addition to the sinusoid, are the *square wave*, the *sawtooth wave*, the *triangle wave*, and a (periodic) *pulse train* of square pulses. See p. 168 for illustrations. (Also fairly common are trapezoidal waves with nonzero rise/fall times, and pulse trains of trapezoidal pulses.)

Some terminology associated with periodic signals:

- *cycle* – One repetition of the signal, whose duration is the period *T*.
- *amplitude* – The signal value on each cycle having the greatest absolute magnitude.
- *peak-to-peak amplitude* – Difference between maximum & minimum signal values.
- *duty cycle* – For a pulse train, % of time the signal is over the halfway point btw. min & max values.
- *pulse width* – For a pulse train, the length of time the signal is over the halfway point between min & max values on each cycle.
- *rise time/fall time* – For a trapezoidal wave or pulse train, time for the signal to go between min & max values.

Sinusoidal waveforms. A sinusoidal signal has the general form:

$$x(t) = A \cdot \sin(\omega t + \phi),$$

where *A* is the amplitude, *ω* the angular frequency, and *φ* the *phase* or phase offset of the signal, in angular units. This can also be related to a *time offset* Δ*t* through:

$$\Delta t = \frac{T}{\text{cyc}} \cdot \frac{1 \text{ cyc}}{2\pi \text{ rad}} \cdot \phi = T \cdot \frac{\phi}{2\pi \text{ rad}} = \frac{T\phi}{2\pi}.$$

Of course, we can also express *φ* in terms of the time offset:

$$\phi = 2\pi \cdot \frac{\Delta t}{T} \text{ (rad)}.$$

To shift between sine and cosine representations at will, we can simply adjust the phase offset by 90°:

$$A \sin(\omega t) = A \cos\left(\omega t - \frac{\pi}{2}\right)$$

$$A\cos(\omega t) = A\sin\left(\omega t + \frac{\pi}{2}\right).$$

Phase offsets can also be defined for non-sinusoidal periodic signals relative to each other, based on their time offsets.

Mean value of a signal. The average or mean value of a periodic signal *x*(*t*) is defined as:

$$\langle x \rangle = \frac{1}{T} \int_{t=0}^{T} x(t)\ \mathrm{d}t.$$

RMS value of a signal. To find the overall strength of a sinusoidal signal or other signal that is symmetric across the *x* axis, the average is not appropriate since it is always 0. Instead, we use the *root-mean-square* or RMS value. It is appropriate for calculating average power. It's defined as:

$$x_{\mathrm{rms}} = \sqrt{\frac{1}{T}\int_{t=0}^{T} x^2(t)\ \mathrm{d}t}.$$

Suppose a periodic signal has an instantaneous current of *i*(*t*). Then the average power dissipation of this current when passing through a resistance of *R* is just:

$$\langle P \rangle = i_{\mathrm{rms}}^2 R.$$

Likewise, if the instantaneous voltage is *v*(*t*), then the power dissipation is $\langle P \rangle = v_{\mathrm{rms}}^2/R$.

The RMS value of a sinusoidal signal is:

$$x_{\mathrm{rms}} = \frac{1}{\sqrt{2}} A$$

where *A* is the signal amplitude. You can easily check this using your calculus.