EEL 3003, INTRODUCTION TO ELECTRICAL ENGINEERING – SUMMER 2013

Supplemental Lecture Notes
Lecture #8 (AC Network Analysis, Part II)
§4.3. Solution of Circuits w. Energy Storage Elements

This is the material that I covered on the whiteboard in class today (Thurs., June 6th).

Consider the following circuit containing a sinusoidal AC voltage source, a resistor, and an unpolarized capacitor. We've labeled all branch voltages and currents, and all node and mesh voltages.

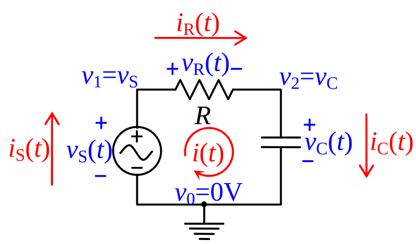

Figure 1. Circuit for AC Solution Example

We of course know immediately from Kirchhoff's Current Law that

$$i_S(t) = i_R(t) = i_C(t) = i. \tag{1}$$

And, from Ohm's Law in the resistor, we know that

$$i_R(t) = \frac{v_R(t)}{R} = \frac{v_1 - v_2}{R} = \frac{v_S - v_C}{R}. \tag{2}$$

From the definition of capacitance, we know that

$$i_C(t) = C\frac{\mathrm{d}v_C}{\mathrm{d}t}, \tag{3}$$

and therefore, since from (1) we know that $i_S = i_C$, we can put (2) and (3) together to get

$$\frac{v_S - v_C}{R} = C\frac{\mathrm{d}v_C}{\mathrm{d}t} \tag{4}$$

or

$$\frac{v_S - v_C}{RC} = \frac{\mathrm{d}v_C}{\mathrm{d}t}. \tag{5}$$

Let's put this into the standard form for a first-order linear ordinary differential equation (ODE), the generic template for which is

$$a\frac{\mathrm{d}y}{\mathrm{d}x} + by + c = 0, \tag{6}$$

where a, b, c are constants, and $y = y(x)$ is an unknown function of the independent variable x. In our case, of course $x = t$ and $y(x) = v_\mathrm{C}(t)$, since the voltage across the capacitor is an unknown function at this point, so, rearranged into this standard form, eq. (5) becomes

$$\frac{\mathrm{d}v_\mathrm{C}}{\mathrm{d}t} + \frac{1}{RC}v_\mathrm{C} - \frac{1}{RC}v_\mathrm{S} = 0. \tag{7}$$

Thus, if the source voltage function $v_\mathrm{S}(t)$ is given, then we can solve eq. (7) for the capacitor voltage function $v_\mathrm{C}(t)$ using standard methods for solving first-order linear ODE's, which you would have learned if you've already had a class on differential equations – but in any case, we'll go through the solution.

First, let's briefly go through an alternative way that we could have set up the problem, using Kirchhoff's Voltage Law instead of KCL. The KVL equation for this circuit can be written as:

$$(v_1 - v_0) + (v_2 - v_1) + (v_0 - v_2) = 0, \tag{8}$$

(this is tautological), or in other words,

$$v_\mathrm{S} - v_\mathrm{R} - v_\mathrm{C} = 0. \tag{9}$$

We can then expand v_R using Ohm's law (2), $v_\mathrm{R} = i_\mathrm{R}R$, and expand v_C using the formula for a capacitor's voltage in terms of the integral of current (see slides from previous lecture), to get

$$v_\mathrm{S} - Ri(t) - \frac{1}{C}\int_{-\infty}^{t} i(\tau)\ \mathrm{d}\tau = 0. \tag{10}$$

A remark: We express the lower limit of the integral as $-\infty$ instead of some definite starting time t_0 because we're about to take a derivative, which will make the starting time irrelevant anyway.

So anyway, differentiate both sides of (10) with respect to time t to get

$$\frac{\mathrm{d}v_s}{\mathrm{d}t} - R\frac{\mathrm{d}i}{\mathrm{d}t} - \frac{1}{C}i(t) = 0. \tag{11}$$

Next, we multiply through by $-1/R$ and rearrange terms to get

$$\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{RC}i - \frac{1}{R}\frac{\mathrm{d}v_s}{\mathrm{d}t} = 0$$

(12)

which is standard first-order linear ODE form, with the unknown function being $i(t)$.

However, rather than solving (12), let's go back to the equation (7) that we derived earlier using KCL, and solve that one instead.

Let's suppose that the source function $v_S(t)$ is given, and that it is sinusoidal. We can, without loss of generality, assume that the signal's phase $\phi = 0$, because the phase can always be made zero through a suitable shift in the time coordinate. Thus, we are left with a source function of the form

$$v_{\mathrm{S}}(t) = V \sin \omega t. \tag{13}$$

Where *V* is the voltage amplitude and *ω* is the angular frequency. We can then substitute this expression back into (7) to get

$$\frac{\mathrm{d}v_{\mathrm{C}}}{\mathrm{d}t} + \frac{1}{RC} v_{\mathrm{C}} - \frac{V}{RC} \sin \omega t = 0 \tag{14}$$

or

$$\frac{\mathrm{d}v_{\mathrm{C}}}{\mathrm{d}t} + \frac{1}{RC} v_{\mathrm{C}} = \frac{V}{RC} \sin \omega t. \tag{15}$$

This is easily solved using standard methods for solving 1$^{\mathrm{st}}$-order linear ODE's, but for those who haven't had a differential equations course, or are rusty, let's review. Equation (15) is similar to the general form

$$a \frac{\mathrm{d}y}{\mathrm{d}x} + by = c \sin x, \tag{16}$$

which can be solved as follows. Consider the family of functions

$$f(x) = a \sin x + b \cos x. \tag{17}$$

These have derivatives

$$f'(x) = a \cos x - b \sin x. \tag{18}$$

Consider now the expression

$$k \frac{\mathrm{d}f}{\mathrm{d}x} + \ell f, \tag{19}$$

where k, ℓ are constants; this expression is designed to match the left-hand side of (16), but renaming *a* to *k* and *b* to *ℓ* to avoid conflict with the *a*, *b* in eqs. (17-18). Substituting the definitions of f and $\mathrm{d}f/\mathrm{d}x$ from (17-18), (19) becomes

$$\begin{aligned} &k(a\cos x - b\sin x) + \ell(a\sin x + b\cos x) \\ &= (\ell a - kb)\sin x + (ka - \ell b)\cos x. \end{aligned} \tag{20}$$

That expression can equal $c\sin x$, the right-hand side of (16), as long as

$$\ell a - kb = c$$
and
$$ka + \ell b = 0. \tag{21}$$

One can then solve these two equations for the unknown constants *a* and *b*, in terms of the known constants *k* and *ℓ*, and plug those back into equation (17) to make that general solution more specific to the particular instance of problem (14) in question.

We'll follow pretty closely that general method in solving (15), except with ωt instead of *x*. Let the solution function $v_\mathrm{C}(t)$ take the general form

$$v_\mathrm{C}(t) = A\sin\omega t + B\cos\omega t, \tag{21}$$

where *A* and *B* are constants to be determined. Substituting this solution template into (15), we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(A\sin\omega t + B\cos\omega t) + \frac{1}{RC}(A\sin\omega t + B\cos\omega t) = \frac{V}{RC}\sin\omega t \\ &(A\omega\cos\omega t - B\omega\sin\omega t) + \frac{1}{RC}(A\sin\omega t + B\cos\omega t) = \frac{V}{RC}\sin\omega t \\ &\left(\frac{A}{RC} - B\omega\right)\sin\omega t + \left(A\omega + \frac{B}{RC}\right)\cos\omega t = \frac{V}{RC}\sin\omega t. \end{aligned} \tag{22}$$

we can see that (22) will be satisfied if the following two equations both hold:

$$\frac{A}{RC} - B\omega = \frac{V}{RC} \tag{23}$$

$$A\omega + \frac{B}{RC} = 0. \tag{24}$$

Solving (24) for *B* gives

$$B = -A\omega RC \tag{24}$$

which when plugged into (23) gives

$$\frac{A}{RC} + A\omega^2 RC = \frac{V}{RC} \tag{25}$$

which when solved for *A* gives

$$A = \frac{V}{1 + (\omega RC)^2} \tag{26}$$

which can then be plugged back into (24) to get

$$B = -\frac{V\omega RC}{1 + (\omega RC)^2}. \tag{27}$$

The equations (26) and (27) can then be plugged back into the general form (21) to now give the complete solution of the differential equation (15):

$$v_{\mathrm{C}}(t) = \frac{V}{1 + (\omega RC)^2} \sin \omega t - \frac{V\omega RC}{1 + (\omega RC)^2} \cos \omega t. \tag{28}$$

This looks complicated, but the important point to bear in mind is that it is ultimately just a weighted sum of sinusoidal functions with constant coefficients and the same underlying frequency $f = \omega/2\pi$. As such, it is itself just a sinusoidal function (we'll prove this later) with the same frequency as the source function $v_{\mathrm{S}}(t)$, but with a different amplitude and phase. In fact, its amplitude in this caseturns out to be just

$$\begin{aligned} V_{\mathrm{C}} &= \sqrt{A^2 + B^2} \\ &= \sqrt{\frac{V^2}{[1 + (\omega RC)^2]^2} + \frac{V^2(\omega RC)^2}{[1 + (\omega RC)^2]^2}} \\ &= \sqrt{\frac{V^2 + V^2(\omega RC)^2}{[1 + (\omega RC)^2]^2}} \\ &= \sqrt{\frac{V^2[1 + (\omega RC)^2]}{[1 + (\omega RC)^2]^2}} \\ &= \sqrt{\frac{V^2}{1 + (\omega RC)^2}} \\ &= \frac{V}{\sqrt{1 + (\omega RC)^2}}. \end{aligned} \tag{29}$$

In any event, the more important point is that this is not just a special case. This example illustrates a more general point, which is that for *any* circuit made of ideal linear elements (resistors, capacitors, inductors), if it's driven only by ideal sinusoidal sources at a single frequency, then it turns out that *every* voltage and current in *the entire circuit* will turn out to also be a sinusoidal function with the very *same* frequency, and just different amplitudes and phases. This observation then allows us to greatly simplify analysis of these circuits, by avoiding generating and solving the entire differential equation for the circuit. Instead, we can just work with more abstract quantities called *phasors* which represent the amplitudes and phases of the different signals in the circuit. But that's a topic for the next lecture.

END OF DOCUMENT