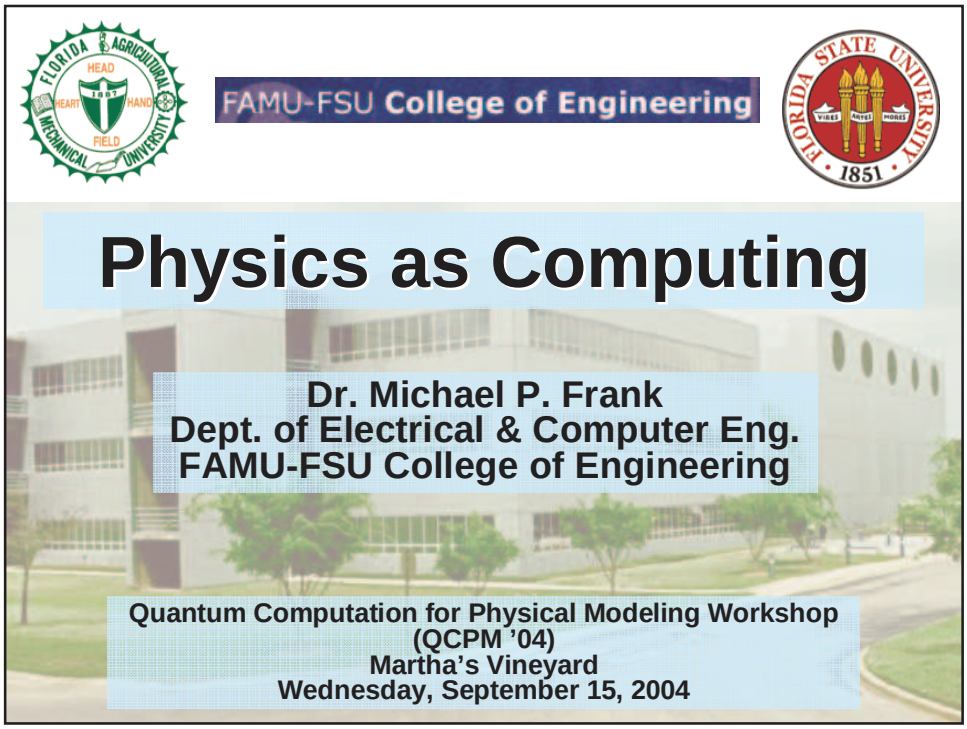


Abstract

- Studying the physical limits of computing encourages us to think about physics in computational terms.
  - Viewing physics "as a computer" directly gives us limits on the computing capabilities of any machine that's embedded within our physical world.
- We want to understand what various physical quantities *mean* in a computational context.
  - Some answers so far:
    - Entropy = Unknown/incompressible information
    - Action = Amount of computational "work"
    - Energy = Rate of computing activity
    - Generalized temperature = "Clock frequency" (activity per bit)
    - Momentum = "Motional" computation per unit distance

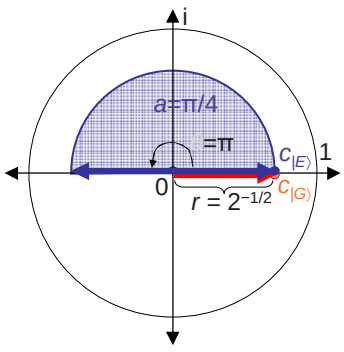
Today's topic

Energy as Computing

- Some history of the idea:
  - Earliest hints can be seen in the original Planck $E=h\nu$ relation for light.
    - That is, an oscillation with a frequency of ν requires an energy at least $h\nu$.
  - Also suggestive is the energy-time uncertainty principle $\Delta E \Delta t \geq \hbar/2$.
    - Relates average energy uncertainty ΔE to minimum time intervals Δt.
  - Margolus & Levitin, *Physica D* 120:188-195 (1998).
    - Prove that a state of *average* energy E above the ground state takes at least time $\Delta t = h/4E$ to evolve to an orthogonal one.
      - Or $(N-1)h/2NE$, for a cycle of N mutually orthogonal states.
  - Lloyd, *Nature* 406:1047-1054, 31 Aug. 2000.
    - Uses that to calculate the maximum performance of a 1 kg "ultimate laptop."
  - Levitin, Toffoli, & Walton, quant-ph/0210076.
    - Investigate minimum time to perform a CNOT + phase rotation, given E.
  - Giovannetti, Lloyd, Maccone, *Phys. Rev. A* 67, 052109 (2003), quant-ph/0210197; also see quant-ph/0303085.
    - Tighter limits on time to reduce fidelity to a given level, taking into account both E and ΔE, amount of entanglement, and number of interaction terms.
- These kinds of results prompt us to ask:
  - Is there some valid sense in which we can say that energy *is* computing?
    - And if so, what is it, exactly?
- We'll see this also relates to *action as computation*.

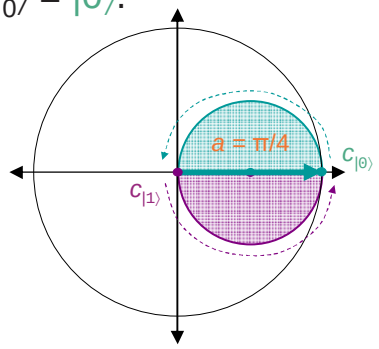
A Simple Example:

- Consider a constant Hamiltonian with energy eigenstates $|G\rangle$ and $|E\rangle$, with eigenvalues $0,E$.
  - That is, $H|G\rangle=0$, $H|E\rangle=E|E\rangle$. *E.g.*, $H = \hbar\Omega\sigma_z$.
- Consider the initial state $|\psi_0\rangle = (|G\rangle+|E\rangle)\cdot 2^{-1/2}$.
  - $c_{|E\rangle}$ phase-rotates at rate $\omega_{|E\rangle} = E/\hbar$.
  - In time $2E/h$, rotates by $\theta=\pi$.
  - The area swept out by $c_{|E\rangle}(t)$ is:
    - $a_{|E\rangle} = \frac{1}{2}\pi(|c_{|E\rangle}|^2) = \pi/4$.
    - This is just ½ of a circle with radius $r_{|E\rangle} = 2^{-1/2}$.
  - Meanwhile, $c_{|G\rangle}$ is stationary.
    - Sweeps out zero area.
  - Total area: $a = \pi/4$.



Let's Look at Another Basis

- Define a new basis $|0\rangle$, $|1\rangle$ with:
  $|0\rangle=(|G\rangle+|E\rangle)\cdot 2^{-1/2}$, $|1\rangle=(|G\rangle-|E\rangle)\cdot 2^{-1/2}$
- Use the same initial state $|\psi_0\rangle = |0\rangle$.
  - Note the final state is $|1\rangle$.
- Coefficients $c_{|0\rangle}(t)$ and $c_{|1\rangle}(t)$ trace out the path shown to the right…
- Note that the total area in this new basis is still $\pi/4$!
  - Area of a circle of radius ½.
- Hmm, is this true for *any* basis? *Yes! …*



Action: Some Terminology

- A physical *action* is, most generally, the integral of an energy over time.
  - Or, along some "temporalizable" path.

$$A = \int_{t=t_1}^{t_2} E(t)\,dt$$

- Typical units of action: h or $\hbar$.
  - Correspond to *angles* of 1 circle and 1 radian, respectively.
- Normally, the word "action" is reserved to refer specifically to the *action of the Lagrangian* L.
  - This is the action in Hamilton's "least action" principle.
- However, note that we can broaden our usage a bit and equally well speak of the "action of" *any* quantity that has units of energy.
  - *E.g.*, the action of the Hamiltonian $H = L + \boldsymbol{p}\cdot\boldsymbol{v} = L + \boldsymbol{p}^2/m$.
- Warning: I will use the word "action" in this more generalized sense!

Action as Computation

- We will argue: *Action* is *computation*.
  - That is, an amount of action corresponds exactly to an amount of physical quantum-computational "work."
    - Defined in an appropriate sense.
- The type of action corresponds to the type of computational work performed, *e.g.*,
  - Action of the Hamiltonian = "All" computational work.
  - Action of the Lagrangian = "Internal" computational work.
  - Action of ***pv*** = "Motional" computational work
- We will show exactly what we mean by all this, mathematically…

Action of the Hamiltonian

- Consider now the action A (eq. (1) below) of any time-dependent Hamiltonian operator $H(t)$.
  - Note that A is an Hermitian observable as well.
- The H determines state-vector dynamics via the usual Schrödinger relation $\mathrm{d}/\mathrm{d}t = \mathrm{i}H/\hbar$.
  - For our purposes, we are adopting the opposite of the usual (but arbitrary) sign convention in this equation.
- This leads to the time-evolution operator (2) below:
  - Given $H(t) \rightarrow A(t_0,t) \rightarrow U(t_0,t)$, any initial vector $v(t_0)$ yields a complete state trajectory $v(t) = U(t,t_0)v(t_0)$.

(1) $$A(t_0,t) = \int_{\tau=t_0}^{t} H(\tau)\,\mathrm{d}\tau$$ (2) $$U(t_0,t) = \mathrm{e}^{\mathrm{i}A(t_0,t)/\hbar}$$

Some Nice Identities for *A*

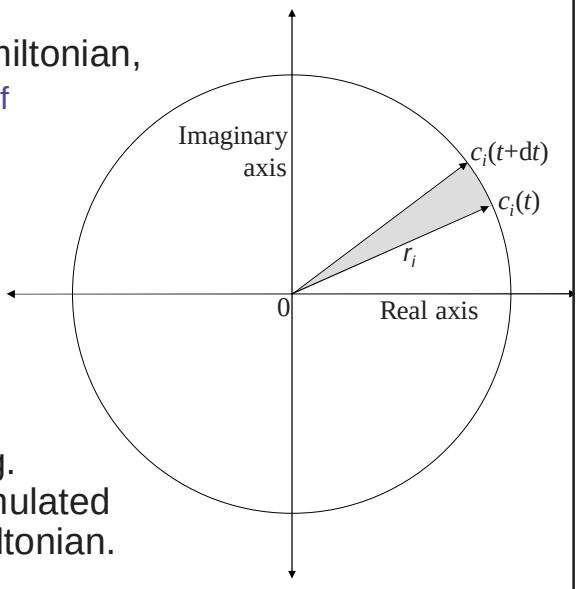
- Consider applying a specific operator A itself to any "initial" state v_0.
  - For any observable A, we'll use shorthand like $Av_0 = \langle v_0|A|v_0\rangle$.
- It is easy to show that Av_0 is equal to all of the following:
  - The quantum-average net phase-angle accumulation of the coefficients c_i of v's components in H's energy eigenbasis $\{v_i\}$, weighted by the component probabilities (3).
  - The line integral, along v's trajectory, of the magnitude of the imaginary part of the inner product $\langle v \mid v + dv \rangle$ between adjacent states (4).
  - Exactly twice the net area a swept out in the complex plane (relative to the complex origin) by v's coefficients c_j, in any basis $\{v_j\}$.
    - We will prove this.
- Note that the value of Av_0 therefore depends *only* on the specific trajectory $v(t)$ that is taken by v_0 itself,
  - and *not* on any other properties of the complete Hamiltonian that was used to implement that trajectory!
    - For example, it doesn't depend on the energy Hu assigned to other states u that are orthogonal to v.

(3) $$Av = \sum_i \Pr[v_i] \int_{\tau=t_0}^{t} \omega_i(t)\,dt$$

(4) $$Av = \int_{v=v_0}^{v_1} \mathrm{Im}\left[\langle v \mid v + dv \rangle\right]$$

Area swept out in energy basis

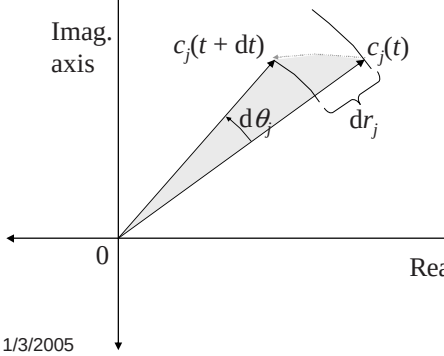
- For a constant Hamiltonian,
  - By a coefficient c_i of an energy basis vector v_i.
- If $r_i=|c_i|=1$, the area swept out is ½ of the accumulated phase angle.
  - For $r_i<1$, note area is this times r_i^2.
- Sum over i = ½ avg. phase angle accumulated = ½ action of Hamiltonian.



In other bases…

- Both the phase and magnitude of each coefficient will change, in general…

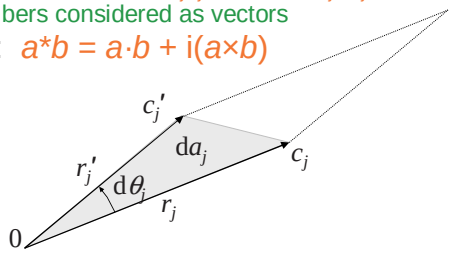
- The area swept out is no longer just a corresponding fraction of a circular disc.



* It's not immediately obvious that the sum of the areas swept out by all the c_j's will still be the same in the new basis.
  - We'll show that indeed it is.

Basis-Independence of a

- Note that each $c_j(t)$ trajectory is just a sum of circular motions…
  - Namely, a linear superposition of the $c_i(t)$ motions…
- Since each circular component motion is continuous and differentiable, so is their sum.
  - The trajectory is everywhere a smooth curve.
    - No sharp, undifferentiable corners.
- Thus, in the limit of arbitrarily short time intervals, the path can always be treated as linear.
  - Area da_j approaches ½ the parallelogram area $r_j r_j' \sin d\theta = c_j \times c_j'$
    - "Cross product" of complex numbers considered as vectors
- Use a handy complex identity: $a^* b = a \cdot b + i(a \times b)$
- Implies that $da_j = \frac{1}{2} \mathrm{Im}[c_j^* c_j']$
  - So, $da = \frac{1}{2} \mathrm{Im}[v^\dagger v']$.
- So da is basis-independent, since the inner product $v^\dagger v'$ is!



Computational Work of a Hamiltonian applied to a system

- Suppose we're given a time-dependent Hamiltonian $H(t)$, a specific initial state v, and a time interval (t_0, t)
  - We can of course compute the operator $A(t_0,t)$ from H.
- We'll call Av the *computational work* performed according to the specific action operator A (or "by" H acting from t_0 to t) on the initial state v.
  - Later we will see some reasons why this identification makes sense.
    - For now, take it as a definition of what we mean by "computational work"
- If we are given only a set V of possible initial vectors,
  - The (maximum, minimum) work of A (or H from t_0 to t) is (5)
- If we had a prob. dist. over V (or equiv., a mixed state ρ),
  - we could instead discuss the *expected* work (6) of A acting on V;

(5)
$$W^+[A] = \max_{v \in V} Av; \quad W^-[A] = \min_{v \in V} Av$$

(6)
$$\overline{W}[A] = \mathrm{Ex}_{v \in V}[Av]$$

Computational "Effort" to Cause a Desired Change

- If we are interested in taking v_0 to v_1, and we have a set $\aleph$ of available action operators A (implied, perhaps, by a set of available Hamiltonians $H(t)$)
  - we define the *minimum work* or "effort" *to get from* v_0 to v_1, (7)
    - Maximizing over $\aleph$ isn't very meaningful, since it may often yield ∞.
- And if we have a desired unitary transform U that we wish to perform on *any* of a set of vectors V, given a set $\aleph$ of available action operators,
  - Then we can define the minimum (over $\aleph$) worst-case (over V) work to perform U, or "worst-case effort to do U" (8).
    - Similarly, we can discuss the best-case effort to do U.
  - or (if we have vector probabilities) the minimum (over $\aleph$) expected (over V) work to do U, or "expected effort to do U" (9).

(7)
$$E[v_0, v_1] = \min_{A \in \aleph:\, v_1 = \exp[iA] v_0} Av_0$$

(8)
$$E^+[U] = \min_{A \in \aleph:\, U = \exp[iA]} W^+[A]$$

(9)
$$\overline{E}[U] = \min_{A \in \aleph:\, U = \exp[iA]} \overline{W}[A]$$

The Justification for All This…

- Why do we insist on referring to these concepts as "computational work" or "computational effort?"
  - One could imagine other possible terms, such as "amount of change," "physical effort," the original "action of the Hamiltonian" *etc.*
    - What is so gosh-darned "computational" about this concept?
- Answer: We can use these concepts to quantify the "size" or "difficulty" of, say, quantum logic-gate operations.
  - And by extension, classical reversible operations embedded in quantum operations…
    - And by extension, classical irreversible Boolean ops, embedded within classical reversible gates with disposable ancillas…
  - As well as larger computations composed from such primitives.
- The difficulty of a given computational op (considered as a unitary U) is given by its *effort* (minimized work over $\aleph$)…
  - We can meaningfully discuss an operation's minimum, maximum, or expected effort over a given space of possible input states.

But, you say, Hamiltonian energy is only defined up to an additive constant…

- Still, the effort of a given U can be a well-defined (and non-negative) quantity, IF…
  - We adopt an appropriate and meaningful zero of energy!
- One useful convention:
  - Define the least eigenvalue (ground state energy) of H to be 0.
    - This ensures that energies are always positive.
- However, we might want to do something different than this in some cases…
  - E.g., if the ground-state energy varies, and it includes energy that had to be explicitly transferred in from another subsystem…
- Another possible convention:
  - We could count total gravitating mass-energy…
- Anyway, let's agree, at least, to just always make sure that all energies are positive, OK?
  - Then the action is always positive, and we don't have to worry about trying to make sense of a negative "amount of computational work."

Energy as Computing

- Given that "Action is computation,"
  - That is, amount of computation,
    - where the suffix "-ation" denotes a noun,
      - i.e., the act itself,
  - What, now, is energy?
- Answer: *Energy is computing*.
  - By which I mean, "rate of computing activity."
    - The suffix "-ing" denotes a verb,
      - the (temporal) carrying out of an action…
- This should be clear, since $H(t) = dA/dt$…
  - Thus the Hamiltonian energy of any given state is the rate at which computational work is being (or would be) performed "on" (or "by," if you prefer) that state.

Applications of the Concept

- How is all this useful?
  - It lets us calculate time/energy tradeoffs for performing operations of interest.
  - It can help us find (or define) lower bounds on the number of operations of a given type needed to carry out a desired computation.
  - It can tell us that a given implementation of some computation is optimal.

Time/Energy Tradeoffs

- Suppose you determine that the effort of a desired $v_1 \rightarrow v_2$ or $U(V)$ (given the available actions $\aleph$) is A.
  - For a multi-element state set V, this could be a minimum, maximum, or expected effort…
- And, suppose the energy that is available to invest in the system in question is at most E.
  - This then tells you directly that the minimum/maximum/expected (resp.) *time* to perform the desired transformation will be $t \geq A/E$.
    - To achieve equality might require varying the energy of the state over time, if the optimal available $H(t)$ says to do so…
- Conversely, suppose we wish to perform a transformation in at most time t.
  - This then immediately sets a scale-factor for the magnitude of the energy E that must be devoted to the system in carrying out the optimal Hamiltonian trajectory $H(t)$; *i.e.*, $E \geq A/t$.

Single-Qubit Gate Scenario

- Let's first look at 2-state (1-qubit) systems.
  - Later we'll consider larger systems.
- Let U be any unitary operator in U_2.
  - *I.e.*, any arbitrary 1-qubit quantum logic gate.
- Let the vector set V consist of the "sphere" of all unit vectors in the Hilbert space $\mathcal{H}_2$.
  - Given this scenario, the *minimum* effort to do any U is always 0 (just let v be an eigenvector of U), and is therefore uninteresting.
    - Instead we'll consider the maximum effort.
- What about our space $\aleph$ of available action operators?
  - Suppose for now, for simplicity, that *all* time-dependent Hermitian operators on $\mathcal{H}_2$ are available as Hamiltonians.
    - Really we only need the time-independent ones, however.
  - Thus, $\aleph$ consists of all (constant) Hermitian operators.

Analysis of Maximum Effort

- The maximum effort to do U (in this scenario) arises from considering a "geodesic" trajectory in U_2.
  - All the worst-case state vectors just follow the "most direct" path along the unit sphere in Hilbert space to get to their destinations.
    - Other vectors "go along for the ride" on the necessary rotation.
- The optimal unitary trajectory $U(t_0,t)$ then amounts to a continuous rotation of the Bloch sphere around a certain axis in 3-space…
  - where the poles of the rotation axis are the eigenvectors of U.
  - Also, there's a simultaneous (commuting) global phase-rotation.
- If we also adopt the convention that the ground-state energy of H is defined to be 0,
  - Then the global phase-rotation factor goes away,
- And we are left with a total effort A that turns out to be exactly equal to $\theta\hbar$, where $0 \le \theta \le \pi$ is simply the (minimum) required angle of Bloch-sphere rotation to implement the given U.

Some Special Cases

- Pauli operators X,Y,Z (including X=NOT), as well as the Hadamard gate:
  - Bloch sphere rotation angle = π (rads)
  - Maximum effort: $h/2$
- Square-root of NOT, also phase gate (square root of Z):
  - Rotation angle $\pi/2$, effort = $h/4$.
- "$\pi/8$" gate (square root of phase gate):
  - Rotation angle $\pi/4$, effort = $h/8$.

Fidelity and Infidelity

- The *fidelity* between pure states u,v is defined as $F(u,v) = |\langle u|v\rangle|$.
  - So, F^2 is the probability of conflating the two.
- Define the *infidelity* between u,v as

$$I(u,v) = \sqrt{1 - F(u,v)^2}$$

- Thus, $I^2 = 1 - F^2$ is the probability that if state u is measured in a basis that includes v as a basis vector, it will project to a basis state other than v.
  - Infidelity is thus a distance metric between states…

Effort Required for Infidelity

- Guess what, a Bloch-sphere rotation by an angle of θ gives a maximum (over V) infidelity of $I^+(\theta) = \sin(2\theta)$.
  - Meanwhile, the minimum fidelity is $\cos(2\theta)$…
    - You'll notice that $F^2+I^2=1$, as probabilities should.
- Therefore, achieving an infidelity of I requires performing a U whose maximum effort is at least $A = 2\hbar\cdot\arcsin(I)$.
  - However, the specific initial states that actually achieve this infidelity under the optimal rotation are Bloch "equator" states
    - Equal superpositions of high and low energy eigenstates;
  - They perform a quantum-average amount of computational work that is only half of the maximum effort.
- Thus, the actual work required for an infidelity of I is only half of the maximum effort, or $W = A/2 = \hbar\cdot\arcsin(I)$.
  - And so, a specific state that carries out an amount of computational work $W \le \pi/2$ can achieve an infidelity of at most $I = \sin(W/\hbar)$, while maintaining a fidelity of at least $F=\cos(W/\hbar)$…
    - a nice simple relation… Especially if we let $\hbar=1$…

Multi-qubit Gates

- Some multi-qubit gates are easy to analyze…
  - E.g., "controlled-*U*" gates that perform a unitary *U* on one qubit only when all of the other qubits are "1"
- If the space of Hamiltonians is truly totally unconstrained, then (it seems) the effort of these will match that of the corresponding 1-bit gates.
  - However, in reality we don't have such fine-tailored Hamiltonians readily available.
- A more thorough analysis would analyze the effort in terms of a Hamiltonian that's expressible as a sum of realistically-available, 1- and 2-qubit controllable interaction terms.
  - We haven't tried to do this yet…

Conclusion

- We can define a clear and stable measure of the "length" of any continuous state trajectory in Hilbert space. (Call it "computational work.")
  - It's simply given by the action of the Hamiltonian.
    - It has a nice geometric interpretation as well.
- From this, we can accordingly define the "size" (or "effort") of any unitary transformation.
  - As the worst-case (or average-case) path length, minimized over the available Hamiltonians.
- We can begin to quantify the effort required for various quantum gates of interest…
  - From this, we can compute lower bounds on the time to implement them for states of given energy.