EEL 3003, INTRODUCTION TO ELECTRICAL ENGINEERING – SUMMER 2013

Lecture Notes
Lecture #13 (Frequency Response & Filters, Part III)
§6.3 (Filters)

This is the material that I covered on the whiteboard in class on Thurs., July 11th.

In previous sections, we (§6.1) defined the frequency response functions such as $H_V(j\omega)$, etc., and (§6.2) discussed how any periodic signal source can be decomposed into a superposition of pure AC sources at various frequency, so that AC circuit analysis tools can be applied more generally to any periodic signal. Now in this section (§6.3), we'll look at how to analyze the (somewhat idealized) frequency response of particular types of 1st-order (RC and RL) and 2nd-order (LRC) circuits, and characterize their behavior as filters.

Recall the general picture of systems including filters for analysis that we introduced in Lecture #11; this is summarized in Figure 1 below. There is a source subcircuit, which could be a pure sinusoidal AC source, or, more generally, any periodic source, which as we saw in the previous lecture can anyway be decomposed into a superposition of a number of (in general, infinitely many) pure sinusoids. This then feeds through a port (two terminals) to the "input" port of a "filter" subcircuit, which (in general, for our purposes) could be any two-port, passive linear circuit. (To say that it's passive means that it has no power sources within it, and to say it's linear means it can be represented in terms of ideal resistors, capacitors, and inductors only.) Finally, the filter feeds through an "output" port to some generic load.

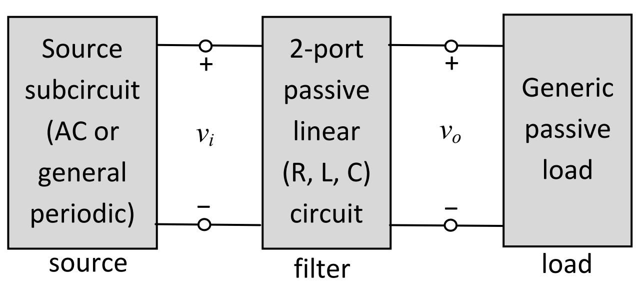


Figure 1. General structure for discussion of filters.

For our purposes in this section, we'll assume the load is passive, and usually that it is high-impedance (so that it draws negligible current) – think of the load, for now, as just being something like a voltmeter, which is just passively measuring the output voltage v_o, not interfering with it at all.

1. Types of Filters

Now, filters may in general come in a variety of types, but for our present purposes, we'll study just four major types: (1) low-pass filters, (2) high-pass filters, (3) bandpass filters, and (4) bandstop (notch) filters. More information about each of these follows.

Each type of filter has a *pass band* – a range of frequencies where most of the signal power is passed through the circuit (exposed at the load) – and a *stop band* – a range of frequencies where most of the signal power is suppressed (or "filtered out") by the response of the filter circuit.

1.1. Low-Pass Filters

A low-pass filter has a frequency-response curve that looks like the one shown in figure 2 below. The horizontal axis is the ratio between the input frequency ω and a designated "cutoff frequency" ω_0, which is defined (by convention) as the frequency at which the frequency response drops to $1/\sqrt{2}$ of its maximum value. Since power scales in proportion to the square of voltage, or to the square of the current, the power delivered is ½ of the input power at the cutoff frequency. Thus, for a low-pass filter, whenever $\omega < \omega_0$, most of the input signal's power is available at the load, whereas when $\omega > \omega_0$, most of the input power is suppressed by the filter. Note that this graph is drawn using a log-log scale, so that you can see that, at high frequencies, the frequency response scales roughly in inverse proportion to the frequency. (Later, we will see the exact analytical form of the response.)

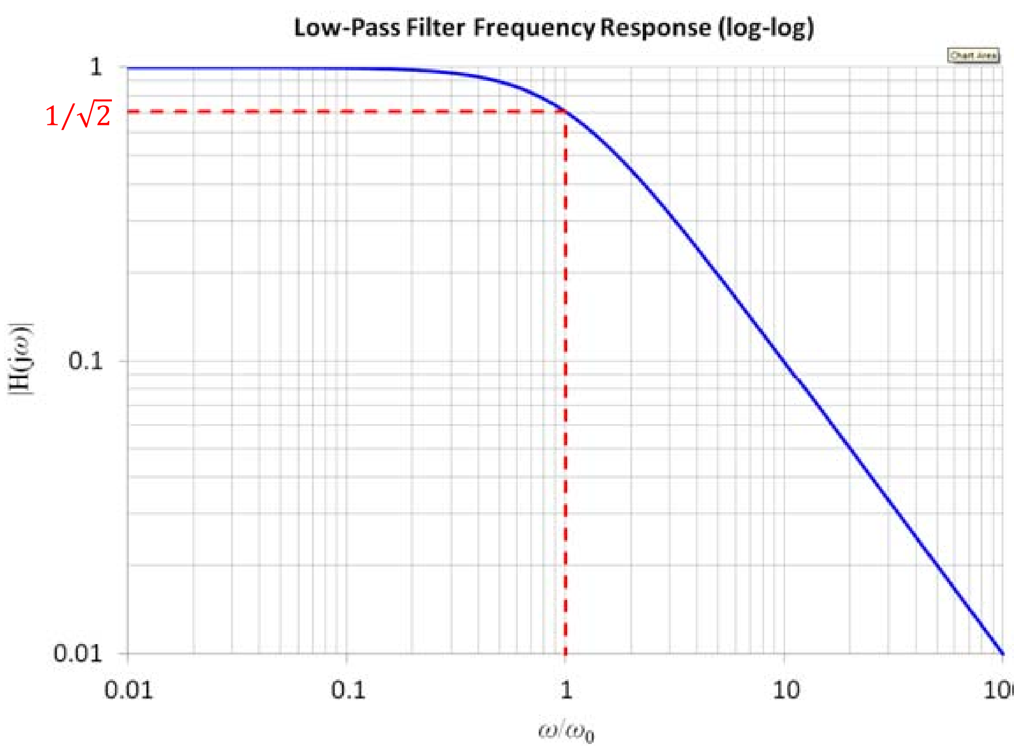


Figure 2. Frequency-response magnitude for a low-pass filter (log-log scale).

1.2. High-Pass Filters

Similarly, the frequency response of a high-pass filter looks like figure 3, below. Again, there is a cutoff frequency ω_0 where the signal amplitude is $2^{1/2}$ as great as the maximum, and power transmitted is ½ of the maximum. However, now the response falls off linearly when $\omega < \omega_0$, and approaches the maximum response when $\omega > \omega_0$, the opposite of the case with the low-pass filter.

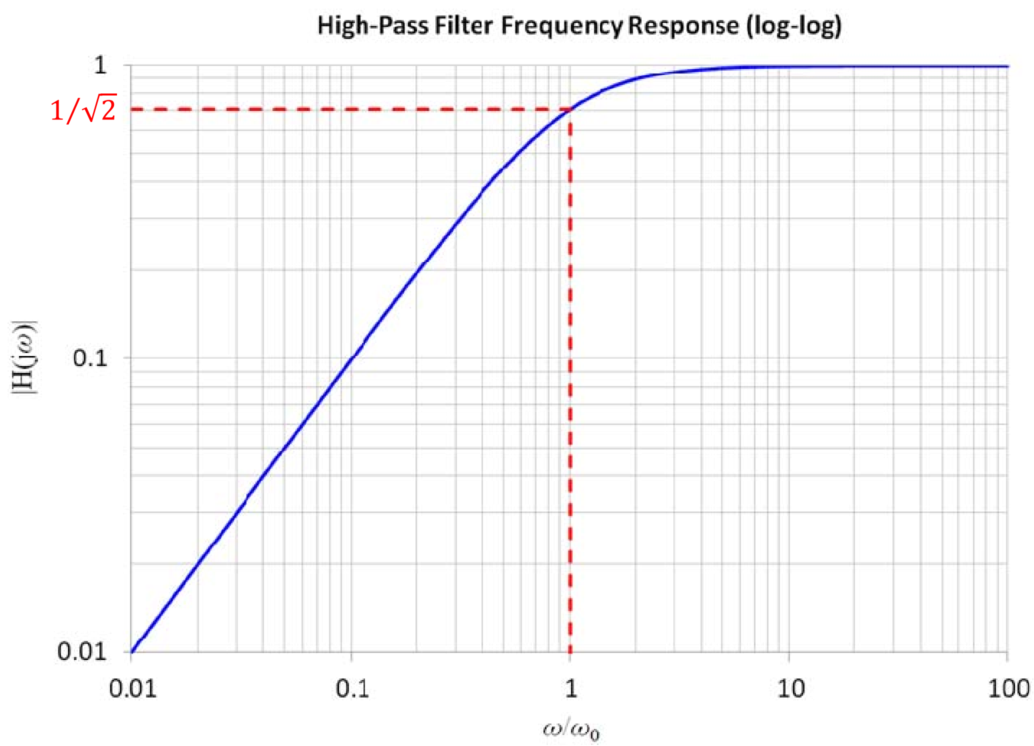


Figure 3. Frequency-response magnitude for a high-pass filter (log-log scale).

1.3. Band-Pass Filters

Band-pass filters (figure 4, below) are so-called because they pass frequencies in a limited band that spans the range in between two different cutoff frequencies, a low cutoff frequency ω_{0L} and a high cutoff frequency ω_{0H}. Halfway between these (on a log scale) is a center frequency ω_C at which the frequency response is maximal (close to 1). The difference between high and low cutoff frequencies is called the *bandwidth* of the filter:

$$B = \omega_{0\mathrm{H}} - \omega_{0\mathrm{H}}. \tag{1}$$

If you study communication theory, you'll learn that the bandwidth of a communication channel is proportional to the rate (in bits per second, say) at which information can be communicated through that channel. We probably won't have time to get into that in this course, but it's good to be aware of it. (For the example band-pass filter shown in figure 4, the bandwidth is 9.9× the center frequency ω_C; in most real narrow-band communication systems (such as a radio station) that ratio is much smaller.)

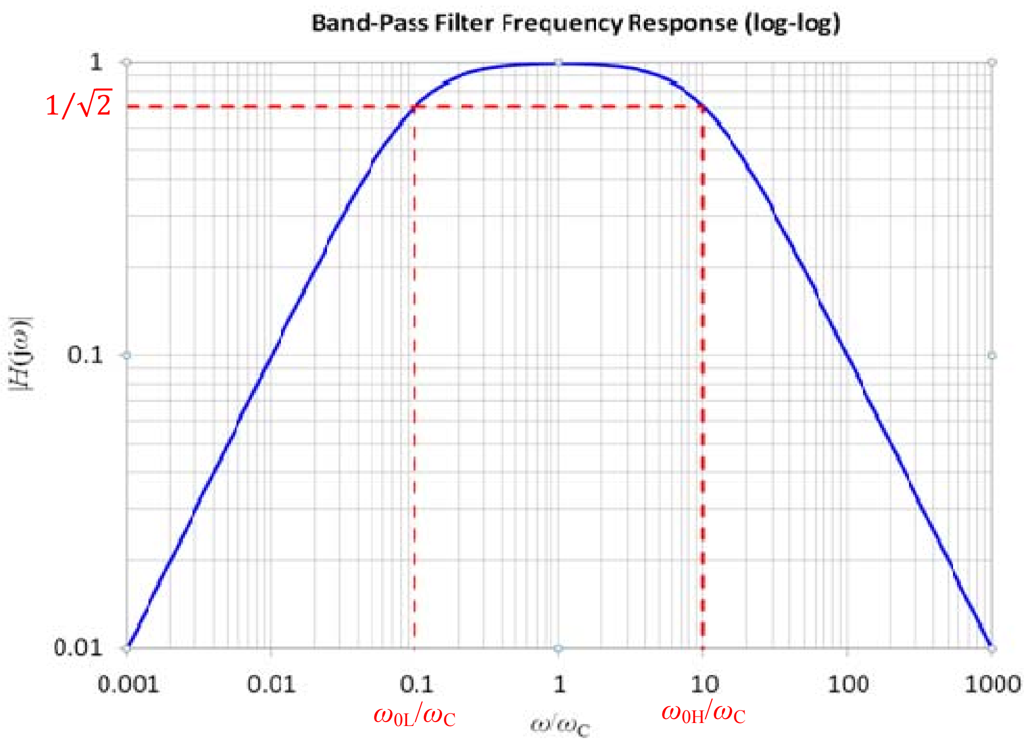


Figure 4. Frequency-response magnitude for a band-pass filter (log-log scale).

A simple band-pass filter can be produced by simply chaining together a low-pass filter and a high-pass filter with appropriate cutoff frequencies, as in figure 5 below, since the overall frequency response is essentially the product of the frequency responses of the two filters; therefore, the overall response will be close to 1 at frequencies where the response of each filter is close to 1.

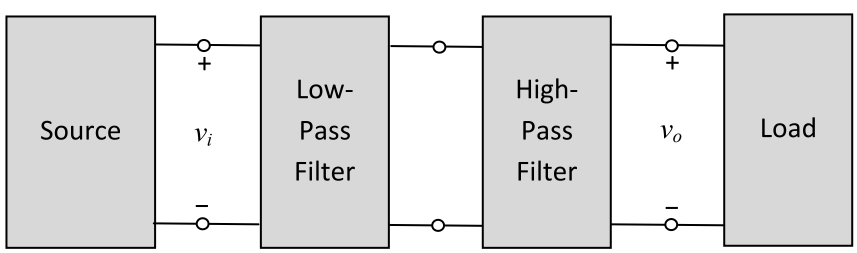


Figure 5. A bandpass filter can be constructed by chaining low- and high-pass filters.

1.4. Band-Stop (Notch) Filters

A band-stop filter is essentially the opposite of a band-pass filter; it passes through low and high frequencies , and suppresses (“stops”) intermediate frequencies (around the center frequency). The region between the cutoff frequencies is called the *stopband*. This is the frequency band where less than half of the input power makes it through the filter.

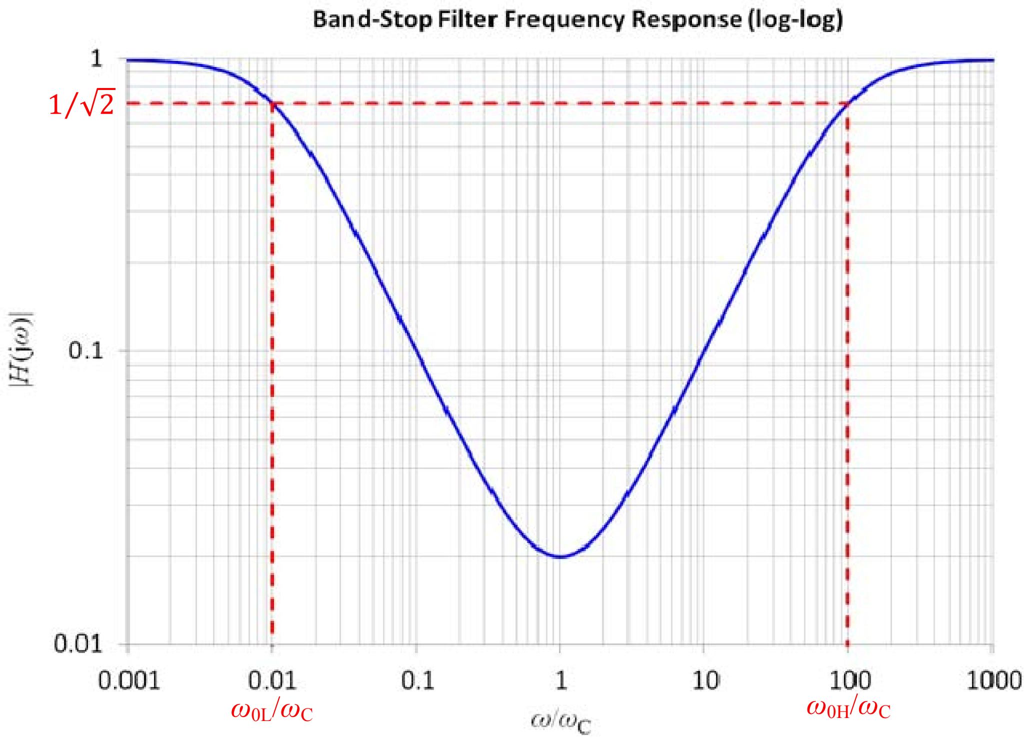


Figure 6. Frequency Response magnitude for a band-stop filter (log-log scale).

You can't create a notch filter by just chaining together a low-pass and a high-pass filter, since if the pass bands of the two filters don't overlap, you won't get a good response at any frequency, and the highest (albeit very low) response of the chained circuit will still be at the center frequency.

However, one simple way to get a notch filter is to construct a circuit that adds together the output signal from a low-band filter in *parallel* with a high-pass filter (with each one using a copy of the same input signal). Adding outputs is easy to do for current signals – just combine the output nodes, and the output currents will automatically add together via KCL. (For voltage signals, it's a little bit more complicated, but we won't worry about that for now.)

2. Simple First-Order Filter Circuits

Low-pass and high-pass filters can be constructed using simple first-order (RC and LR) circuits; in this section, we'll present some basic first-order filter circuit structures and analyze them.

2.1. Low-Pass Filter Circuits

An example of a first-order low-pass filter structure is the simple RC filter shown in figure 7 below. Please note that this consists of a resistor in *series* with the source (and load) terminals and a capacitor *parallel* to the source and load terminals. This is important. From your basic circuit knowledge, it should be easy to see why this is a low-pass circuit. Low frequency means, at frequencies approaching DC (direct or constant current). You know that in a steady-state situation, with a DC voltage applied to it, a capacitor does not carry any current (since if it did, the voltage across it would be changing, not constant). Therefore, in this situation, all of the current from the source enters the load, since it is blocked from crossing the capacitor. In the case of a high-impedance load, the current is small, so the voltage drop across *R* is small, so the voltage at the load approaches 100% of the source voltage, and the power transferred to the load approaches 100% of the power transmitted by the source. Thus, this is a low-pass filter, since the frequency response approaches 1 at low (approaching DC) frequencies.

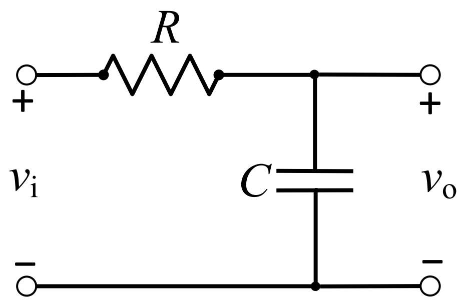


Figure 7. Basic RC low-pass filter structure.

What about at higher frequencies? Assuming the load is relatively high-impedance, so that negligible current passes through it, we can apply the voltage-division rule (in the AC domain, that is, using complex impedances instead of resistances) to the left-hand loop (passing from the "+" input terminal through *R* and *C* to the "–" input terminal), giving us:

$$H_V(j\omega) = \frac{\mathbf{V}_o(j\omega)}{\mathbf{V}_i(j\omega)} = \frac{Z_C}{Z_R + Z_C} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}. \quad (2)$$

What is the magnitude of $H_V(j\omega)$? Using the general rule that for complex z, $|1/z| = 1/|z|$, we have

$$|H_V(j\omega)| = \left|\frac{1}{1 + j\omega RC}\right| = \frac{1}{|1 + j\omega RC|} = \frac{1}{\sqrt{1 + (\omega RC)^2}}. \quad (3)$$

What about the phase of $H_V(j\omega)$? We can apply the arctangent rule here,

$$\angle H_V(j\omega) = \text{atan}\frac{\text{Im}[H_V(j\omega)]}{\text{Re}[H_V(j\omega)]}, \quad (4)$$

but first we have to find the real and imaginary parts of $H_V(j\omega)$:

$$\begin{aligned} H_V(j\omega) &= \frac{1}{1 + j\omega RC} \cdot \frac{1 - j\omega RC}{1 - j\omega RC} = \frac{1 - j\omega RC}{1 + (\omega RC)^2} = \\ &= \frac{1}{1 + (\omega RC)^2} - j\frac{\omega RC}{1 + (\omega RC)^2} \\ \therefore \quad \text{Re}[H_V(j\omega)] &= \frac{1}{1 + (\omega RC)^2}, \quad \text{Im}[H_V(j\omega)] = \frac{-\omega RC}{1 + (\omega RC)^2}, \end{aligned} \quad (5)$$

therefore

$$\frac{\mathrm{Im}[H_\mathrm{V}(\mathrm{j}\omega)]}{\mathrm{Re}[H_\mathrm{V}(\mathrm{j}\omega)]} = \frac{\left[\frac{-\omega RC}{1+(\omega RC)^2}\right]}{\left[\frac{1}{1+(\omega RC)^2}\right]} = -\omega RC, \tag{6}$$

so

$$\angle H_\mathrm{V}(\mathrm{j}\omega) = \mathrm{atan}(-\omega RC) = -\,\mathrm{atan}(\omega RC). \tag{7}$$

Altogether, in polar form, putting together (3) and (7), we have that the voltage frequency response is:

$$H_\mathrm{V}(\mathrm{j}\omega) = \frac{1}{\sqrt{1+(\omega RC)^2}} \cdot \mathrm{e}^{-\mathrm{j}\,\mathrm{atan}(\omega RC)}. \tag{8}$$

Now, let's find the cutoff frequency ω_0. Recall, this is defined as the frequency at which the magnitude of the frequency response drops to $1/\sqrt{2}$ of its maximum value. The maximum magnitude of H_V is clearly 1, since

$$\lim_{\omega\to 0}|H_\mathrm{V}(\mathrm{j}\omega)| = \lim_{\omega\to 0}\frac{1}{\sqrt{1+(\omega RC)^2}} = \frac{1}{\sqrt{1}} = 1. \tag{9}$$

(Contrariwise, as $\omega \to \infty$, $|H_\mathrm{V}(\mathrm{j}\omega)| \to 0$.) Thus, we want to find the value of ω_0 such that

$$|H_\mathrm{V}(\mathrm{j}\omega_0)| = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{1+(\omega_0 RC)^2}}. \tag{10}$$

Clearly, this is only true when

$$\begin{aligned} &2 = 1 + (\omega_0 RC)^2 \\ &(\omega_0 RC)^2 = 1 \\ &\omega_0 RC = 1 \\ &\omega_0 = \frac{1}{RC}. \end{aligned} \tag{11}$$

Therefore, the term ωRC in equations like (2), (3), (7), and (8) can be rewritten as ω/ω_0; this gives what's known as the *canonical* (*i.e.*, standard) form of the frequency response for a first-order low-pass filter:

$$H_\mathrm{V}(\mathrm{j}\omega) = \frac{1}{1+\mathrm{j}\left(\frac{\omega}{\omega_0}\right)} = \left[\frac{1}{1+(\omega/\omega_0)^2}\right] - \mathrm{j}\left[\frac{\omega/\omega_0}{1+(\omega/\omega_0)^2}\right]$$

$$= \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}} \angle - \operatorname{atan} \frac{\omega}{\omega_0}. \tag{12}$$

Other first-order filter structures (besides the simple RC filter of Figure 7) have different values for ω_0, but otherwise share the same canonical form as (12).

Figure 8 below (from textbook fig. 6.17) illustrates the amplitude (magnitude) and phase of the voltage frequency response for the RC low-pass filter. In this example, $\omega_0 = 1$. Note that in this figure, unlike in figure 2, the vertical scale is linear, not logarithmic. Therefore, the amplitude of the frequency response can be seen to asymptotically approach 0 (its logarithm approaching $-\infty$) as ω goes to infinity. The phase of the frequency response, meanwhile, asymptotically approaches $-\operatorname{atan}(\infty) = -90° = -\pi/2$.

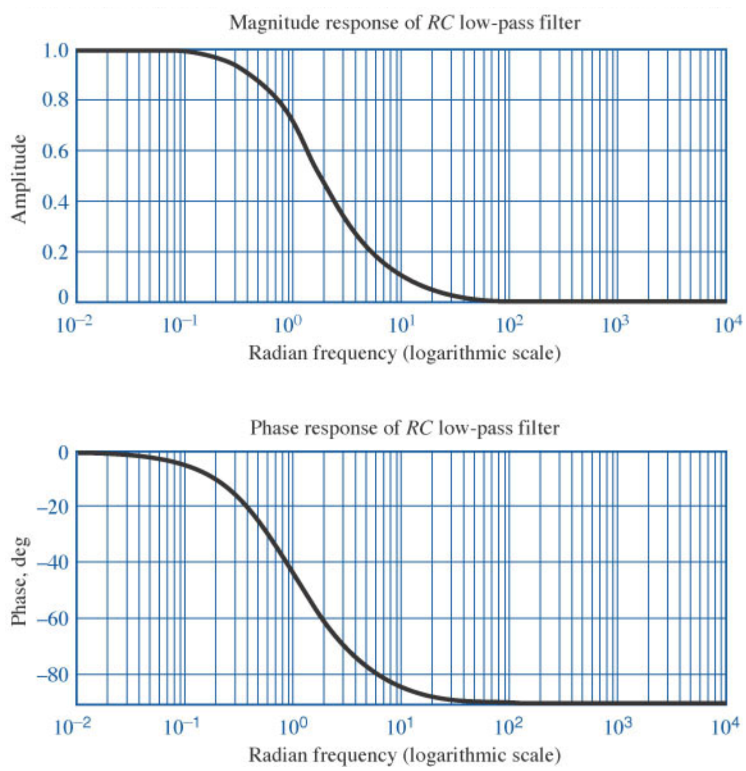


Figure 8. Example frequency response magnitude and phase for an RC low-pass filter.

Now, you can also make a first-order low-pass filter using an inductor and a resistor, instead of a capacitor and a resistor. Simply substituting the inductor for the capacitor won't work, since, as you know, an inductor behaves differently from a capacitor. But, as you might be able to guess, if you switch things around, and put the inductor in *series* with the source, and put the resistor in parallel in the source, like in figure 9 below, that works.

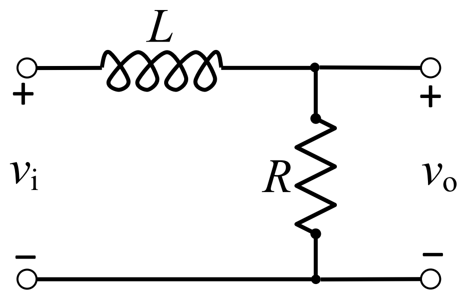


Figure 9. Basic LR low-pass filter structure.

Let's analyze the frequency response for that circuit, using the same assumption as before, of a high-impedance load:

$$H_V(j\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_i} = \frac{R}{R + Z_L} = \frac{R}{R + j\omega L} = \frac{1}{1 + j\omega(L/R)}. \tag{13}$$

This now matches the exact same canonical form (12) as the RC low-pass filter, that is,

$$H_V(j\omega) = \frac{1}{1 + j\left(\frac{\omega}{\omega_0}\right)} = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}} \angle -\operatorname{atan}\frac{\omega}{\omega_0}, \tag{14}$$

except this time with the cutoff frequency being

$$\omega_0 = \frac{R}{L}. \tag{15}$$

So, in terms of *R* and *L*, the magnitude and phase of H_V are:

$$\begin{aligned} |H_V(j\omega)| &= \frac{1}{\sqrt{1 + (\omega L/R)^2}}, \\ \angle H_V(j\omega) &= -\operatorname{atan}(\omega L/R). \end{aligned} \tag{16}$$

2.2. High-Pass Filter Circuits

It turns out that to create a simple first-order high-pass filter circuit, all you have to do is swap the resistive and reactive elements in the low-pass filter circuits of Figures 7 and 9. Thus, the following are high-pass filter structures:

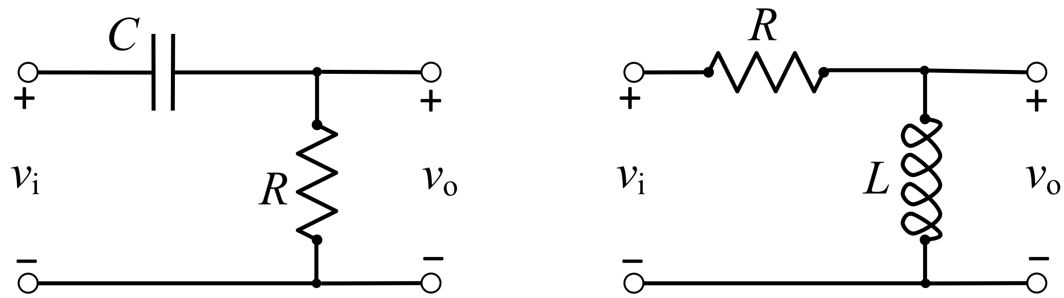


Figure 10. Basic RC and LR high-pass filter structures.

Let's take a quick look at the analysis of the RC high-pass filter in a high-impedance load scenario:

$$H_{\mathrm{V}}(\mathrm{j}\omega) = \frac{\mathbf{V}_{\mathrm{o}}}{\mathbf{V}_{\mathrm{i}}} = \frac{R}{R + Z_C} = \frac{R}{R + \frac{1}{\mathrm{j}\omega C}} = \frac{\mathrm{j}\omega RC}{1 + \mathrm{j}\omega RC}, \tag{17}$$

thus

$$|H_{\mathrm{V}}(\mathrm{j}\omega)| = \frac{|\mathrm{j}\omega RC|}{|1 + \mathrm{j}\omega RC|} = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}}. \tag{18}$$

Setting this equal to $1/\sqrt{2}$, we find the cutoff frequency:

$$|H_{\mathrm{V}}(\mathrm{j}\omega_0)| = \frac{1}{\sqrt{2}} = \frac{\omega_0 RC}{\sqrt{1 + (\omega_0 RC)^2}}$$

$$\therefore\ \frac{1}{2} = \frac{(\omega_0 RC)^2}{1 + (\omega_0 RC)^2}$$

$$\therefore\ 1 + (\omega_0 RC)^2 = 2(\omega_0 RC)^2$$

$$\therefore\ (\omega_0 RC)^2 = 1$$

$$\therefore\ \omega_0 RC = 1$$

$$\therefore\ \omega_0 = \frac{1}{RC},$$

the same as in the case of the RC low-pass filter. The difference now is that, for the high-pass filter, we approach the maximum magnitude of frequency response when $\omega > \omega_0$, in the limit as $\omega \to \infty$:

$$\lim_{\omega\to\infty} |H_{\mathrm{V}}(\mathrm{j}\omega)| = \lim_{\omega\to\infty} \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}} = \lim_{\omega\to\infty} \frac{\omega RC}{\sqrt{(\omega RC)^2}} = \lim_{\omega\to\infty} \frac{\omega RC}{\omega RC} = 1, \tag{19}$$

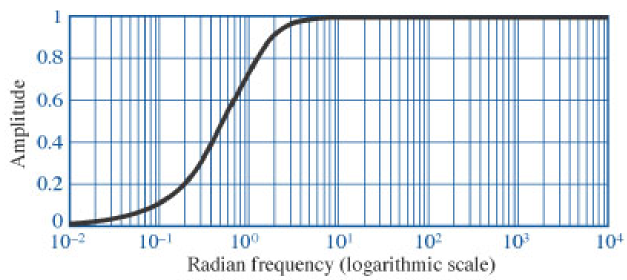
and the minimum frequency response of 0 is approached as when $\omega < \omega_0$, in the limit as $\omega \to 0$.

Similarly, if you compute the cutoff frequency for the LR high-pass filter, you'll find that you'll have $\omega_0 = R/L$, the same as for the LR low-pass filter.

As you can easily verify, the canonical form for the frequency response of a generic first-order (either RC or LR) high-pass filter is:

$$\begin{aligned} H_{\mathrm{V}}(\mathrm{j}\omega) &= \frac{\mathrm{j}(\omega/\omega_0)}{1+\mathrm{j}(\omega/\omega_0)} \\ &= \frac{\omega/\omega_0}{\sqrt{1+(\omega/\omega_0)^2}} \angle \frac{\pi}{2}\mathrm{rad} - \mathrm{atan}\frac{\omega}{\omega_0}. \end{aligned} \tag{20}$$

Example curves showing the magnitude and phase of the frequency response for a first-order high-pass filter are shown in Figure 11 below. Note that these are similar to the curves for the low-pass filter (Figure 8) except that they transition in the opposite direction as the frequency increases.



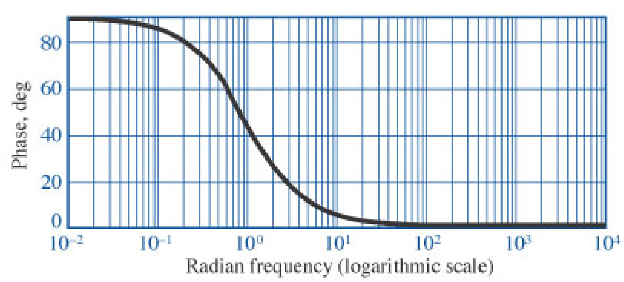


Figure 11. Example frequency response magnitude and phase for an RC high-pass filter.

Please note that the above analysis of first-order filters was all done within a simplified scenario wherein we pretended that the load had infinite (real) impedance, *i.e.*, is an open circuit. A more realistic example of a circuit where the load has only a finite impedance, which must then be taken into account in the analysis of the filter's frequency response, is given in example 6.8 in the textbook, which you should review on your own time.

3. Second-Order Filter Circuits

Second-order circuits are required to construct bandpass and notch filters. In lecture we briefly covered a simple example of a 2nd-order (LRC) bandpass filter, namely the circuit shown in Figure 12 below:

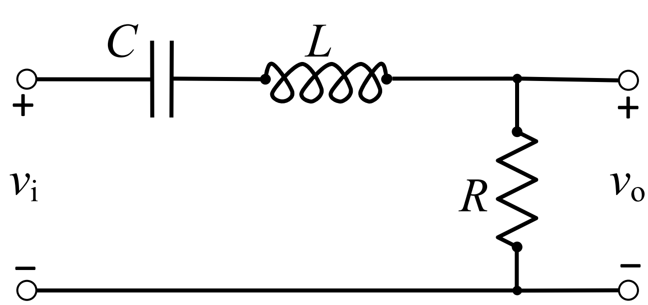


Figure 12. Basic second-order LRC bandpass filter structure.

The basic form of the voltage frequency response for this circuit (in the high-*Z* load scenario) is:

$$H_{\mathrm{V}}(\mathrm{j}\omega) = \frac{\mathbf{V}_{\mathrm{o}}}{\mathbf{V}_{\mathrm{i}}} = \frac{R}{Z_L + R + Z_C} = \frac{R}{\mathrm{j}\omega L + R + \frac{1}{\mathrm{j}\omega C}} = \frac{\mathrm{j}\omega RC}{(\mathrm{j}\omega)^2 LC + \mathrm{j}\omega RC + 1}, \tag{21}$$

and its canonical form is

$$\begin{aligned} H_{\mathrm{V}}(\mathrm{j}\omega) &= \frac{(2\zeta/\omega_{\mathrm{n}})\mathrm{j}\omega}{\left(\mathrm{j}\frac{\omega}{\omega_{\mathrm{n}}}\right)^2 + \left(\frac{2\zeta}{\omega_{\mathrm{n}}}\right)\mathrm{j}\omega + 1} \\ &= \frac{\left({}^{1}\!/_{Q\omega_{\mathrm{n}}}\right)\mathrm{j}\omega}{\left(\mathrm{j}\frac{\omega}{\omega_{\mathrm{n}}}\right)^2 + \left({}^{1}\!/_{Q\omega_{\mathrm{n}}}\right)\mathrm{j}\omega + 1}, \end{aligned} \tag{22}$$

where we have defined the following key quantities:

- The natural or resonant frequency

$$\omega_{\mathrm{n}} = \sqrt{\frac{1}{LC}}, \tag{23}$$

- The quality factor

$$Q = \frac{1}{2\zeta} = \frac{1}{\omega_{\mathrm{n}} RC} = \frac{1}{R}\sqrt{\frac{L}{C}}, \tag{24}$$

- The damping ratio

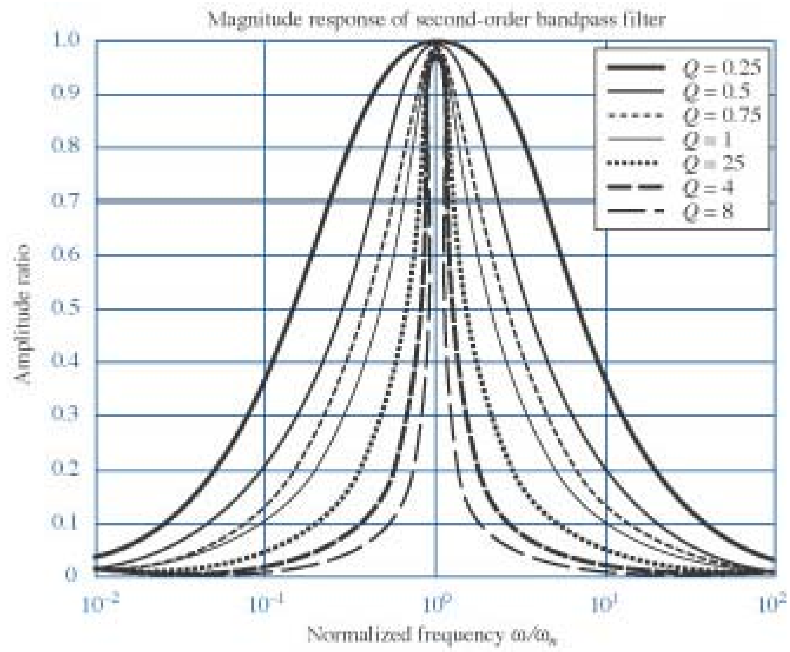
$$\zeta = \frac{1}{2Q} = \frac{R}{2}\sqrt{\frac{C}{L}}, \tag{25}$$

- The ½-power bandwidth

$$B = \frac{\omega_{\mathrm{n}}}{Q}. \tag{26}$$

The bandwidth *B* is the same as the one we introduced in eq. (1). In communication applications, it relates to the maximum rate at which information can be conveyed through a communication channel.

Finally, figure 12 below illustrates the frequency response of a 2^{nd}-order bandpass filter for a range of quality-factor values. As you can see (and as implied by eq. 26), as the quality factor increases, the bandwidth becomes more narrow, which is appropriate for purposes of (for example) selecting for a particular frequency (or narrow-band channel) in the input signal. High-quality LRC circuits therefore find utility in wireless communication applications.



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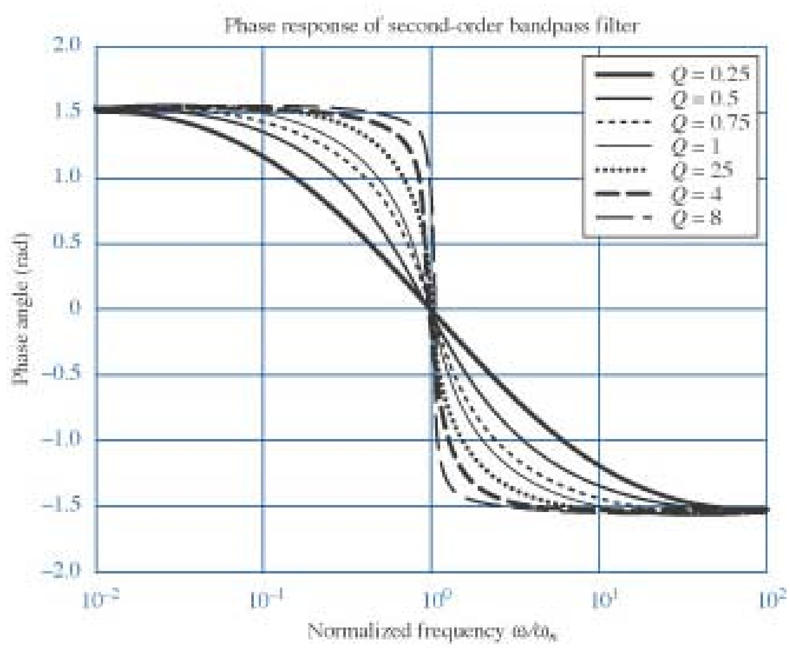

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