Research Memo

Energy Transfer and Recovery Efficiencies for Adiabatic Charging with Various Driving Waveforms

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Abstract

A standard problem that arises in the study of energy-recovering circuits for digital logic is to analyze the relative energy efficiency of different possible voltage waveforms for the power/clock signals that are used to drive adiabatic charge transfers in the logic. To well characterize these waveform efficiency issues provides an analytical tool that is helpful in the design optimization of efficient power/clock signal generators. In this memo, we define appropriate metrics for the comparison of the energy efficiencies of different voltage waveforms in the context of fixed requirements on the energy transferred to the logic load or the energy supplied by the power/clock resonator, and show how these efficiency metrics scale as functions of frequency for a variety of simple wave shapes in a basic lumped-element circuit model.

1 Introduction

[Need more citations throughout this section.]

The semiconductor industry is well aware that conventional *irreversible* digital logic technology, which dissipates the entire energy of a digital signal to heat upon each manipulation of its digital value, cannot be pushed many orders of magnitude beyond present levels of energy efficiency [?, ?]. Eventually, the only way to make further progress in the energy efficiency of digital technologies will

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be to increasingly utilize energy-recovering, highly adiabatic techniques for modifying digital signal values in an asymptotically thermodynamically reversible fashion. With continued refinement, such techniques can in principle eventually offer energy efficiencies that are many orders of magnitude beyond the limits of conventional irreversible technology, making possible systems having much higher total digital throughput within realistic power dissipation constraints than would otherwise be possible.

Today, a number of (hopefully temporary!) roadblocks still conspire to prevent the widespread development of reversible, adiabatic logic concepts into a practical, commercializable technology for high power-performance computing:

- Lack of a complete and correct understanding of the principles of adiabatic switching and reversible logic among much of the low-power design community;
- Presence of substantial design flaws in most of the existing published literature on adiabatic circuits in CMOS, leading to an underestimation, in most of the papers and review articles on the subject, of the energy efficiency that can be achieved by proper adiabatic design;
- A real lack of detailed designs and demonstrations of high-quality powersupply clock resonators suitable for driving adiabatic logic with high systemlevel energy efficiency;
- A per-device manufacturing cost today that, although much smaller than it was in the past, makes the hardware overheads of adiabatic, reversible approaches currently still somewhat prohibitive in terms of the extra hardware cost that is required to achieve better than conventional levels of throughput within system power constraints.

Items 1 and 2 are educational issues that we expect can be more easily solved after there have been initial empirical demonstrations of working, practical designs that use sound adiabatic design principles to outperform conventional designs, and the construction of such demonstration systems is one of our most important research goals.

Item 4 is an issue that we expect will naturally solve itself over time, as competitive pressures (driven especially by memory applications) continue to force down per-device costs to the point where logic power dissipation rather than logic cost becomes an ever-increasingly more dominant constraint on achievable digital throughput. This is already beginning to happen today, as logic designers working in the latest technologies find they are constrained more and more by power consumption rather than by device count.

This leaves item 3 as the primary area where concrete technical progress needs to be made in the near term; having good resonators is a prerequisite for building empirical demonstrations that will be sufficiently impressive to silence the critics, show the world what can be done with good adiabatic design, and thereby begin to solve problems 1 and 2. A detailed study of the issues and problems that exist in the area of highquality resonator design for adiabatic logic therefore needs to be undertaken.

One early step in this research program is for us to thoroughly understand and characterize the impact on energy efficiency of the precise shape of the voltage waveform that is delivered to the logic by the power/clock resonator. Sinusoidal waveforms are the simplest ones to generate, but they are general unsuitable for driving high-quality CMOS-based adiabatic logic, which requires for maximum energy efficiency that the driving signal must remain flat for a substantial portion of the clock period in order to allow sufficient time for devices to be turned off without simultaneously trying to drive a current through them (an act which would lead to nonadiabatic dissipation [?]). However, alternative devices for adiabatic logic based on quantized states, such as the Quantum-Dot Cellular Automata (QDCA) devices being studied by Notre Dame and Sandia, do not suffer from the same requirement for flat-topped waveforms, and can be appropriately driven by a more general class of waveforms, including sinusoidal ones, while remaining fully adiabatic.

However, regardless of the precise constraints on the waveform shape that may or may not be imposed by the requirements of a particular switching technology, any implementation of reversible computing whose adiabatic transitions are driven by a time-varying voltage source will involve charge transfer onto some kind of capacitive load along some resistive path.

For example, in CMOS, the load is a set of transistor gates along with various parasitic node capacitances, while the resistive charging path consists of the effective resistance of the transistor channel along with other parasitic resistances in interconnects, *etc..*; whereas in QDCA, the load consists of an array of fairly well-isolated single-electron quantum dots that are capacitively coupled to their immediate surroundings, while the resistive charging path consists of the parasitic resistances that exist along whatever wires are being used for distribution of the clock signal that drives and controls the timing of the logic transitions within the dots. QDCA may ultimately have higher performance than CMOS, due in part to the fact that the driving clock signal does not need to be gated through any transistors at all, while only single electrons within the QDCA devices are required to pass through the relatively more resistive tunnel junctions between the dots.

But regardless of the details, in either technology (CMOS or QDCA), we can roughly partition our model of the system into the elements that are driving the resonant oscillation of the clock signal (e.g., some inductors or transmission lines) and the parasitic loading elements that make up the logic system that is being driven by that signal.

For any system that makes use of clocked adiabatic charge transfers, the question naturally arises as to the amount of energy dissipation that arises as the result of those charge transfers. Furthermore, one wishes to characterize how the dissipation changes as a function of various design parameters, including not only the frequency of operation but also the clock waveform shape and the magnitudes of the resistances and capacitances (whether necessary or parasitic) that make up the load. Only with a thorough understanding of these basic issues



Figure 1: Circuit schematic for initial power dissipation modeling.

can we begin to design and optimize complete adiabatic logic systems consisting of a clock oscillator resonantly driving adiabatic transitions of a logic load, and to do this job with the confidence that we are analyzing and optimizing our designs correctly.

The goal of the present memo is simply to solidify our own understanding of these issues by doing a thorough job of analyzing the energy efficiency of charge transfers in the context of a simple, general lumped-element circuit model. Later research along this same line of work will expand the scope of our analyses by encompassing more detailed circuit models that capture more accurately the physical behavior of realistic systems, including more detailed and realistic models of resonators and clock-distribution networks, together with logic models that characterize more accurately the circuit-level characteristics of particular categories of devices such as CMOS transistors and QCA cells.

The contents of this memo are as follows. Section 2 presents our basic, general circuit model. Section 3 defines our notations for a variety of basic energy and power quantities associated with charge transfers in this circuit. Sec. 4 defines a couple of important metrics of energy efficiency that we focus on, motivated by the twin desires to achieve high power-performance in the logic as well as high quality factors in the power-clock resonators. Then sections 5-9 proceed to carry out detailed analyses of energy efficiency for particular categories of driving waveforms. Section 10 summarizes our results and suggests directions for follow-on work. Finally, Appendix A gives several detailed derivations of the classic $CV^2/2$ formula for energy transferred onto the load, while Appendix B details one way to solve the particular differential equation for the load voltage that is encountered in §9.

2 Basic circuit model

As a simple starting point for analysis, in this memo we will use the basic lumped-element RC circuit model illustrated in figure 1. The load (a set of devices to be driven) is modeled as a load capacitance of C in series with the resistance R of the charging path. The clock generator is modeled as a general AC voltage source producing a (not necessarily sinusoidal) voltage waveform $v_s(t)$,



Figure 2: Illustration of example source and load voltage waveforms, with various notations defined. The rising part of the $v_{\rm s}(t)$ source voltage trajectory is just any arbitrary monotonically non-decreasing function, while the falling part traverses the identical trajectory but in the negative direction. This "voltage reversal symmetry" property helps simplify the analysis. The v(t) load curve illustrated is the actual steady-state solution (numerically estimated) that follows from the given $v_{\rm s}(t)$ for a certain value of RCf. For purposes of quantifying energy transferred onto the load, we use $V_{\rm min}$ as our reference voltage, while for quantifying energy supplied and recovered by the source, we use $V_{\rm s,min}$. The difference between these two voltage references is given by $V_{\rm gap} = V_{\rm min} - V_{\rm s,min}$. In some analyses, we may also temporarily use the voltage reference $V_{\rm avg}$ for convenience.

with maximum voltage $V_{\rm s,max}$, minimum voltage $V_{\rm s,min}$, and a time-averaged voltage $V_{\rm avg}$ which we assume is equal to $(V_{\rm s,max} + V_{\rm s,min})/2$. The amplitude of the signal is defined as $V_{\rm sa} = V_{\rm s,max} - V_{\rm avg} = (V_{\rm s,max} - V_{\rm s,min})/2$. In response to the voltage signal applied to the source terminal, a time-dependent current i(t) will flow through the resistor, while the voltage on the load capacitor will vary according to some function of time v(t), with amplitude $V_{\rm a} \leq V_{\rm sa}$ and total swing $V = 2V_{\rm a} \leq V_{\rm s}$. For resistance R > 0 and frequency f > 0, the load voltage in general experiences some amount of amplitude damping and phase lag $\theta_{\rm lag}$ relative to the driving signal, as illustrated in figure 2.

Using this basic circuit model, we will analyze a variety of important cases to determine the energy dissipation and the energy efficiency according to various metrics. The analyses we carry out in subsequent sections are for the most part fairly standard and well-known in the adiabatic circuits literature, but we repeat them here for pedagogical purposes, as well as to better develop our own understanding of these results.

3 General power and energy analysis

In this section, we show how to derive expressions for various important power and energy quantities relating to the above circuit starting from the most basic principles of circuit theory.

First, let's build up our analytical model of the circuit dynamics. From the definition of capacitance, we have

$$C = \mathrm{d}q/\mathrm{d}v,\tag{1}$$

where q(t) is the instantaneous charge stored on the "upper plate" of the capacitance as a function of time. Meanwhile, from the definition of current and the fact that injected charge builds up on a capacitor, we have that the instantaneous current in the indicated direction through the resistor is

$$i = \mathrm{d}q/\mathrm{d}t.\tag{2}$$

Finally, Ohm's Law gives us that the instantaneous current is also

$$i = \Delta v/R. \tag{3}$$

where $\Delta v(t) = v_{\rm s}(t) - v(t)$ is the voltage drop across the resistor. Combining eqs. 1-3 and solving for dv/dt, we obtain the ordinary differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \Delta v/RC. \tag{4}$$

Thus, in general for the above circuit, given any input waveform $v_{s}(t)$, the instantaneous voltage v(t) on the load must obey the equation (4), which can also written as

$$RC\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \Delta v. \tag{5}$$

In later sections, we will examine the solutions of this ODE for a variety of particular cases of interest.

In general, assuming that we have obtained a solution for v(t), then by Ohm's law again, the instantaneous current i(t) is given by

$$i = \frac{\Delta v}{R}.$$
(6)

Now, the definition of the voltage v of a circuit node is

$$v = \frac{\mathrm{d}e}{\mathrm{d}q} \tag{7}$$

where e(q) is the total electrostatic potential energy contained on that circuit node, which is a function of the quantity q of charge contained on it. Thus, v dq = de. The definition of the power flowing into a given node is p = de/dt = v dq/dt = vi.

It is important to remember that voltages v, energies e, and power flows p are all generally only defined up to an additive constant; thus, these quantities cannot be expressed numerically; only differences between them can. **[So?]**

For a current flowing across a resistor, the power flowing into the source terminal of the resistor at voltage v_s is $v_s i$, whereas the power flowing out the other terminal at its voltage v is vi; the difference $v_s i - vi = i(v_s - v) = i\Delta v$ between these is thus the instantaneous amount of power that is dissipated by the current flow in the resistor,

$$p_{\rm d} = i\Delta v = \frac{\Delta v^2}{R}.$$
(8)

To determine the total energy dissipated in the resistor between any two given times, say 0 and τ , one then merely integrates $p_{\rm d}$ over t:

$$E_{\rm d} = \int_{t=0}^{\tau} p_{\rm d}(t) \,\mathrm{d}t = \int_{0}^{\tau} \frac{[\Delta v(t)]^2}{R} \mathrm{d}t.$$
(9)

The average power dissipation over this period is then given simply by $P_{\rm d} = E_{\rm d}/\tau$. Of course, one should keep in mind that in general, there may be additional dissipation in the unknown circuit that implements the waveform generator for the voltage source $v_{\rm s}(t)$. However, eq. 9 remains satisfactory as far as the power dissipation directly associated with the resistance R is concerned.

In later sections of this document, we will analytically derive closed-form expressions for $E_{\rm d}$ and $P_{\rm d}$ in a variety of important basic cases. But first, some general statements about the energy flow are in order.

For voltage sources that alternate between a monotonic increase through the entire voltage range of $V_{\rm s}$ over time t, followed by a polarity-reversal-symmetric monotonic decrease backwards through the same range in the same amount of time, the energy flow in the circuit over the course of this cycle can be summarized as illustated in figure 3. Meanwhile, figure 4 shows the detailed time-evolution of the cumulative energy flow for the particular source and load waveforms illustrated in figure 2.

Relative to the reference voltage $V_{\rm s,min},$ the total energy supplied by the source during charging of the load is

$$E_{\rm sup} = \int_{t=\tau_0}^{\tau_1} \left[v_{\rm s}(t) - V_{\rm s,min} \right] \cdot i(t) \,\mathrm{d}t \tag{10}$$

where τ_0 and τ_1 are the times at which the current flow in the forwards direction starts and stops, respectively, as illustrated in figure 2.

The total energy dissipated in the resistance during this low-to-high transition of the load is

$$E_{\rm d,tr} = \int_{t=\tau_0}^{\tau_1} p_{\rm d}(t) \,\mathrm{d}t = \int_{t=\tau_0}^{\tau_1} \Delta v(t) i(t) \,\mathrm{d}t, \tag{11}$$



Figure 3: Schematic summary of per-cycle energy flow given source waveforms that alternate between monotonic increase and symmetric monotonic decrease over the range $V_{\rm s}$. The gap energy $E_{\rm gap} = Q_{\rm tfr} V_{\rm gap}$ represents the apparent decrease in the amounts of energy supplied, transferred to the load, and recovered that arises if one moves from measuring energies in the source-based reference frame $V_{\rm s,min} = 0$ to the load-based frame $V_{\rm min} = 0$. By our convention, $E_{\rm sup}$ and $E_{\rm rec}$ are defined relative to the former, while $E_{\rm tfr}$ is defined relative to the latter. These conventions affect our later definitions of the energy transfer and energy recovery efficiencies.



Figure 4: Detailed temporal evolution of the cumulative energy transferred for the specific source and load waveforms illustrated in figure 2. The important energy quantities that are the focus of this memo are indicated.

while the remaining part of E_{sup} is the energy transferred onto the load capacitance (in the base- $V_{s,min}$ reference frame), and can be expressed as

$$E_{\rm sup} - E_{\rm d,tr} = \int_{t=\tau_0}^{\tau_1} \left[v_{\rm s}(t) - V_{\rm s,min} \right] i(t) \, \mathrm{d}t \tag{12}$$

$$-\int_{t=\tau_0}^{\tau_1} \left[v_{\rm s}(t) - v(t) \right] i(t) \,\mathrm{d}t \tag{13}$$

$$= \int_{t=\tau_0}^{\tau_1} [v(t) - V_{s,\min}] i(t) dt$$
 (14)

$$= \int_{t=\tau_0}^{\tau_1} \left[v(t) - V_{\min} + V_{gap} \right] i(t) \, \mathrm{d}t \tag{15}$$

$$= \left\{ \int_{\tau_0}^{\tau_1} \left[v(t) - V_{\min} \right] i(t) \, \mathrm{d}t \right\} + V_{\mathrm{gap}} \int_{\tau_0}^{\tau_1} i(t) \, \mathrm{d}t \qquad (16)$$

$$= E_{tfr} + V_{gap}Q_{tfr}$$
(17)
$$= E_{tfr} + E_{res}$$
(18)

$$= E_{\rm tfr} + E_{\rm gap}, \tag{18}$$

where in (15) we are using the gap voltage $V_{\text{gap}} = V_{\text{min}} - V_{\text{s,min}}$ from figure 2, and in (17) we are using E_{tfr} to represent the "useful" part of the energy transferred, which, for our purposes in this document, is defined as

$$E_{\rm tfr} = \int_{t=\tau_0}^{\tau_1} \mathrm{d}e_{\rm tfr}(t) \tag{19}$$

$$= \int_{t=\tau_0}^{\tau_1} [v(t) - V_{\min}] \, \mathrm{d}q(t)$$
 (20)

$$= \int_{t=\tau_0}^{\tau_1} \left[v(t) - V_{\min} \right] \frac{\mathrm{d}q}{\mathrm{d}t} \,\mathrm{d}t \tag{21}$$

$$= \int_{t=\tau_0}^{\tau_1} \left[v(t) - V_{\min} \right] \, i(t) \, \mathrm{d}t, \qquad (22)$$

where we have defined the differential energy transfer with reference to the minimum load voltage $V_{\rm min}$, via the relation $de_{\rm tfr} = [v(t) - V_{\rm min}] \, \mathrm{d}q$. In (17) we are also defining and using the total charge transferred $Q_{\rm tfr} = CV$, and in (18) we are using the "energy gap" $E_{\rm gap} = V_{\rm gap}Q_{\rm tfr}$, which is needed to compensate for the fact that $E_{\rm sup}$ and $E_{\rm tfr}$ are defined using different base voltages. The overall relation $E_{\rm sup} - E_{\rm d,tr} = E_{\rm tfr} + E_{\rm gap}$ is pictured in figure 3.

Appendix A shows that, given the definition (22), it follows that the energy transferred also obeys the simple relation

$$E_{\rm tfr} = \frac{1}{2}CV^2,\tag{23}$$

and explains in more detail how the gap energy E_{gap} arises.

Now, by symmetry considerations in the AC steady state, the energy dissipated during the subsequent high-to-low transition is also $E_{d,tr}$, just as in the low-to-high transition, and so the total energy dissipated during both transitions over the course of the complete charge/discharge cycle is

$$E_{\rm d,cyc} = 2E_{\rm d,tr}.\tag{24}$$

Since in a sustained steady-state cycle, the energy dissipated per cycle intuitively (and by the first law of thermodynamics) cannot be greater than the total energy supplied by the source during a cycle, we have that

$$E_{\rm d,cyc} \le E_{\rm sup}.$$
 (25)

We note that this constraint also implies an upper bound on $E_{d,tr}$:

$$E_{\rm d,tr} = E_{\rm d,cyc}/2 \tag{26}$$

$$\leq E_{\sup}/2.$$
 (27)

If $E_{d,cyc} < E_{sup}$, then the remaining part of E_{sup} is returned to the AC voltage source terminal, where it may potentially be recovered by (for example) a resonant power-clock oscillator for reuse on subsequent cycles; the amount of potentially recoverable energy is thus given by

$$E_{\rm rec} = E_{\rm sup} - E_{\rm d,cyc} \tag{28}$$

$$= E_{\rm sup} - 2E_{\rm d,tr} \tag{29}$$

$$= E_{\rm tfr} + E_{\rm gap} - E_{\rm d,tr} \tag{30}$$

$$\geq 0,$$
 (31)

where the inequality in the last line follows from (28) and (25), and implies that always

$$E_{\rm tfr} + E_{\rm gap} \ge E_{\rm d,tr}.$$
 (32)

An important caveat to note at this point is that our choice of measuring $E_{\rm sup}$ and $E_{\rm rec}$ relative to $V_{\rm s,min}$ while measuring $E_{\rm tfr}$ relative to $V_{\rm min}$ is in some sense completely arbitrary; really, any voltage reference could have been chosen, and so each of these energies is really only physically meaningful up to an arbitrary additive constant, whose magnitude depends on the quantity $Q_{\rm tfr} = CV$ of charge transferred and on where we choose to set our zero of voltage and thus of electrostatic energy. For example, another possible choice of definitions that would still satisfy the constraint (25) would involve always setting $E_{\rm rec} = 0$ and redefining $E_{\rm tfr} = E_{\rm sup} - E_{\rm d,tr}$, which would imply that $E_{\rm tfr} = E_{\rm d,tr} = E_{\rm sup}/2$ by eqs. 28-31, with $E_{\rm gap}$ eliminated from eq. 30. However, this particular choice would then teach us nothing about the relative efficiencies of different charging profiles.

Our particular choice of voltage conventions can be fairly well justified by the fact that for purposes of charging up a given load through a given voltage swing, only C and V are given, and so the energy transfer should depend only on these, whereas when evaluating the energy recovery efficiency of a resonant circuit, it is the source voltage swing V_s that matters, and so the most natural, interesting question to ask in that context is how much of the energy supplied, relative to the base of the swing, is returned to the source.

Another point is that in the low-frequency limit which is our primary regime of operation in adiabatic systems, $V_{\min} \rightarrow V_{s,\min}$, so the two different voltage references that we are using turn out to be nearly the same as each other anyway. But, it is important to keep in mind that at higher frequencies, the difference between these voltages is larger, and this fact can affect the energy transfer efficiency substantially, and in ways that have practical impact, for example by requiring a relatively larger voltage swing on the source in order to cause a given desired energy transfer onto the load.

We should also warn the reader that, in some of our analyses, we may temporarily choose a voltage reference of $V_{\text{avg}} = 0$, rather than either of the abovementioned choices, in order to simplify the mathematical expressions for the waveforms. However, it should be understood that when this is done, it is merely a temporary choice, and we always shift our results back to our conventional voltage references for purposes of characterizing the energy efficiency quantities that are of interest to us, which we'll now define.

[Note: I still need to finish propagating all of my new voltage/energy conventions throughout all of the remaining sections.]

4 Energy efficiency metrics

For purposes of choosing a waveform shape that minimizes energy dissipation in the context of fixed requirements on the energy usefully transferred to the logic and the frequency of charge/discharge cycles, we define and will make use of a key figure of merit which we call the *energy transfer efficiency*,

$$\eta_{\rm E,tfr} = \frac{E_{\rm tfr}}{E_{\rm d,tr}},\tag{33}$$

which is simply the amount of energy that is usefully transferred onto the load during charging, expressed as a multiple of the amount of energy that is dissipated during this process.

As we'll show in the next section [move to earlier], in general for a constant load capacitance C that is taken through a total low-to-high voltage range of V, we have that the energy transferred onto the load, relative to V_{\min} , is exactly

$$E_{\rm tfr} = \frac{1}{2}CV^2\tag{34}$$

so that in general equation 33 becomes

$$\eta_{\rm E,tfr} = \frac{CV^2/2}{E_{\rm d,tr}} \tag{35}$$

$$= \frac{CV^2}{E_{\rm d,cyc}}.$$
 (36)

Note that this particular efficiency metric ranges from $(0, \infty)$ rather than from (0, 1); we could have $\eta_{\text{E,tfr}} < 1$ if the energy transfer is so inefficient that more energy is dissipated per load transition than is actually going onto or off of the load (which we'll see is what occurs at high frequencies), whereas we could have $\eta_{\text{E,tfr}} > 1$ if less energy is dissipated than is transferred.

In subsequent sections we'll derive the exact form of eq. 36 as a function of frequency for various specific wave shapes.

For some purposes, such as when assessing the quality factor Q for a resonator system driving a logic load, it may also be of interest to characterize the energy recovery efficiency

$$\eta_{\rm E,rec} = \frac{E_{\rm rec}}{E_{\rm sup}} \tag{37}$$

$$= \frac{E_{\rm tfr} + E_{\rm gap} - E_{\rm d,tr}}{E_{\rm tfr} + E_{\rm d,tr}}$$
(38)

$$= \frac{(\eta_{\rm E,tfr} - 1)E_{\rm d,tr} + E_{\rm gap}/E_{\rm d,tr}}{(\eta_{\rm E,tfr} + 1)E_{\rm d,tr}}$$
(39)

$$= \frac{\eta_{\rm E,tfr} - 1 + E_{\rm gap}/E_{\rm d,tr}}{\eta_{\rm E,tfr} + 1}$$
(40)

which is the fraction of energy supplied by the source that is recovered for reuse on subsequent cycles. Note that for low-frequency driving signals that allow sufficient time between transitions for the load voltage v(t) to converge exponentially close to $V_{\rm s,max}$, the energy $E_{\rm gap}$ becomes completely negligible and so (40) simplifies to

$$\eta_{\rm E, rec} \to \frac{\eta_{\rm E, tfr} - 1}{\eta_{\rm E, tfr} + 1}.$$
(41)

Finally, we if define the *quality factor* Q of the charge/discharge process to be the ratio between energy supplied by the source and energy dissipated during the cycle,

$$Q = \frac{E_{\rm sup}}{E_{\rm d,cyc}},\tag{42}$$

then we can express Q and $\eta_{\rm E, rec}$ in terms of each other as

$$Q = \frac{E_{\rm sup}}{E_{\rm sup} - E_{\rm rec}} = \frac{E_{\rm sup}}{E_{\rm sup} - \eta_{\rm E, rec} E_{\rm sup}}$$
(43)

$$= (1 - \eta_{\rm E, rec})^{-1}, \tag{44}$$

$$\eta_{\rm E, rec} = 1 - Q^{-1}, \tag{45}$$

so for example if a given circuit and driving waveform combination provides $\eta_{\rm E,rec} = 0.999$, then it offers a Q factor of 1,000.

Incidentally, one should keep in mind that this Q represents the energyrecovering quality of the process of charging and discharging the load by itself; it does not take into account any additional inefficiencies that may exist in the waveform generator circuitry. Thus the overall Q of a complete system will in general be somewhat less than that described here. Later memos in this series will show in detail how to calculate an overall system Q factor for particular classes of resonator designs.

This concludes our general circuit analysis; now we go on to analyze specific classes driving waveforms to determine how $\eta_{\rm E,tfr}$ and $\eta_{\rm E,rec}$ vary as functions of frequency in each instance.

5 Step-function charging

First, assume that the voltage source $v_s(t) = 0$ for all t < 0 and $v_s(t) = V_s$ for all $t \ge 0$. This then implements a step up in voltage by V_s at time 0. The solution of (5) in this case is v(t) = 0 for t < 0, and

$$v(t) = V_{\rm s} \left(1 - \mathrm{e}^{-t/RC} \right) \tag{46}$$

for $t \ge 0$. The current is

$$i = \frac{V_{\rm s}}{R} \mathrm{e}^{-t/RC},\tag{47}$$

and the power dissipation is

$$p = \frac{V_{\rm s}^2}{R} \mathrm{e}^{-2t/RC}.$$
(48)

The total energy dissipated over $t \in (-\infty, \infty)$ is

$$E_{\rm d} = \int_{t=0}^{\infty} \frac{V_{\rm s}^2}{R} \mathrm{e}^{-2t/RC} \mathrm{d}t \tag{49}$$

$$= \frac{V_{\rm s}^2}{R} \int_0^\infty \mathrm{e}^{-2t/RC} \mathrm{d}t \tag{50}$$

$$= \frac{V_{\rm s}^2}{R} \frac{RC}{-2} e^{-2t/RC} \Big|_{t=0}^{\infty}$$
(51)

$$= -\frac{1}{2}CV_{\rm s}^2 \left(e^{-\infty} - e^0 \right)$$
 (52)

$$= \frac{1}{2}CV_{\rm s}^2. \tag{53}$$

Note that this classic formula for the energy dissipation that results from charging a load by direct connection to a voltage source with a constant voltage of $V_{\rm s}$ does not depend at all on the magnitude of R. In fact, R can even have an arbitrary time-dependence without changing the result, as is demonstrated by another derivation of the same formula that is obtained even more directly.

Consider that a total charge of $Q_{\text{tfr}} = CV$ must be moved onto the capacitor in order to charge it up by $V = V_s$; the energy supplied by the source and carried initially by this amount of charge when it leaves the source at voltage of V (relative to $V_{s,\min} = v(0) = 0$) is $E_{\sup} = Q_{\text{tfr}}V = CV^2$, while the total electrostatic energy transferred to and accumulated on the capacitor during charging, relative to the initial v(0) = 0, is

$$E_{\rm tfr} = \int dE_{\rm tfr} = \int_{q=0}^{Q_{\rm tfr}} \left[v(q) - v(0) \right] \, dq = \int_{q=0}^{Q_{\rm tfr}} \frac{q}{C} \, dq = \frac{Q_{\rm tfr}^2}{2C} = \frac{1}{2}CV^2.$$
(54)

The remaining energy of

$$E_{\rm sup} - E_{\rm tfr} = CV^2 - \frac{1}{2}CV^2 = \frac{1}{2}CV^2$$
(55)

is not stored anywhere in the circuit, so it must be dissipated in the charging process. Thus we find the same answer $E_{sup} = CV^2/2$ in a way that is clearly completely independent of the precise shape of the load voltage trajectory v(t), and that depends only on the fact that the source voltage is a constant V throughout the entire charging process.

Note that in the case of a step function, we can see that the energy transfer efficiency is

$$\eta_{\rm E,tfr} = \frac{E_{\rm tfr}}{E_{\rm d,tr}}$$
(56)

$$= \frac{CV^2/2}{CV^2/2}$$
(57)

$$=$$
 1. (58)

Thus, using step functions at asymptotically zero frequency, the amount of energy that is usefully transferred at each step (that is, the amount moved onto or off of the load) is equal to the amount that is dissipated.

Meanwhile, since in this limit $E_{\text{gap}} = V_{\min} - V_{\text{s,min}} = 0$, the energy recovery efficiency over a complete cycle can be derived from (41) as

$$\eta_{\rm E,rec} = \frac{\eta_{\rm E,tfr} - 1}{\eta_{\rm E,tfr} + 1} = 0/2$$
(59)

$$= 0.$$
 (60)

That is, none of the energy supplied by the source is recovered at the end of the cycle for an asymptotically zero-frequency square wave; instead, all of it is dissipated; the first half during charging, and the second half during discharging.

6 Square wave driver

The previous section applies to sources that are square waves with frequency approaching zero. We now do an exact analysis for a source waveform that is a square wave toggling between voltages $V_{\rm s,min} = 0$ and $V_{\rm s} = V_{\rm s,max}$ with a 50% duty cycle and an arbitrary finite, non-infinitesimal frequency of f. (Figure 5.) Let $\tau = 1/f$ be the clock period, thus $\tau/2 = 1/2f$ is the time between clock



Figure 5: Source and load waveforms for a square-wave driver with clock period $\tau = 8RC$. In this figure, the time and voltage intervals are scale-free, in arbitrary units.

edges. Let a given rising edge be at t = 0 when the load voltage is at $v(0) = v_0$, which will in general be greater than $V_{s,\min}$ in the steady state (not equal because the load voltage doesn't have time to decay all the way to $V_{s,\min}$). Thus, before time t = 0, v will have been still falling, whereas after the rising edge, v will then begin rising, so also we have that $V_{\min} = v_0$ and $V_{gap} = v_0 - V_{s,\min} = v_0$. For the first half-cycle, the load response initially looks exactly as it would for a step-function source rising by an amount $V_s - v_0$ (from v_0 to $V_{s,\max}$) at time 0; *i.e.*, for $t \in [0, \tau/2]$, we can apply an appropriately scaled and shifted version of eq. 46, namely

$$v(t) = v_0 + (V_{\rm s} - v_0) \left(1 - e^{-t/RC}\right).$$
(61)

A bit later, at the time $t = \tau/2$, the source voltage then steps back down to 0 and a similar (and symmetric) discharge process begins. Due to symmetry considerations, the steady-state load waveform must be an even function vertically centered on the $v = V_{\text{avg}} = V_{\text{s}}/2$ axis, so we must have $v(\tau/2) = V_{\text{s}} - v_0$. Thus we have the equation

$$v(\tau/2) = V_{\rm s} - v_0$$
 (62)

$$v_0 + (V_{\rm s} - v_0) \left(1 - e^{-\tau/2RC} \right) = V_{\rm s} - v_0$$
 (63)

$$v_0 + V_{\rm s} - V_{\rm s} {\rm e}^{-\tau/2RC} - v_0 + v_0 {\rm e}^{-\tau/2RC} = V_{\rm s} - v_0 \tag{64}$$

$$v_0 \left(1 + \mathrm{e}^{-\tau/2RC} \right) = V_{\mathrm{s}} \mathrm{e}^{-\tau/2RC} \tag{65}$$

$$v_0 = \frac{V_{\rm s} {\rm e}^{-\tau/2RC}}{1 + {\rm e}^{-\tau/2RC}},$$
 (66)

which implies that the magnitude of the initial voltage drop across the resistor immediately after each (rising or falling) clock edge is

$$|\Delta V_0| = V_{\rm s} - v_0 = \frac{V_{\rm s}}{1 + e^{-\tau/2RC}}.$$
(67)

Using now (51) with $|\Delta V_0|$ in place of V_s and $t = \tau/2$ in place of $t = \infty$ gives, for the energy dissipation during each half-cycle,

$$E_{\rm d,tr} = \frac{\Delta V_0^2}{R} \frac{RC}{-2} e^{-2t/RC} \Big|_{t=0}^{\tau/2}$$
(68)

$$= \frac{1}{2}C(\Delta V_0)^2(1 - e^{-\tau/RC}).$$
 (69)

Note that at low frequencies $f \to 0$, $\Delta V_0 \to V_s$, and so this expression approaches $\frac{1}{2}CV_s^2$ which, as we already saw in the previous section, is the dissipation for charging a load all the way through a step of $V = V_s$ using a constant-voltage source.

Over the course of a full cycle the total energy dissipation $E_{\rm d,cyc}$ is twice that of (69), and so in terms of the cycle frequency $f = 1/\tau$, the average power dissipation is

$$P_{\rm d} = C(\Delta V_0)^2 f(1 - e^{-1/RCf}).$$
(70)

Note that when $\tau \ll RC$, $E_{\rm d} \to 0$ because there is not time to dissipate much energy, but in this same limit, the voltage swing V on the load of

$$V = (V_{\rm s} - v_0) - v_0 = V_{\rm s} - 2v_0 =$$
(71)
$$V_{\rm s} - \frac{\tau}{2RC}$$

$$= V_{\rm s} - 2 \frac{V_{\rm s} {\rm e}^{-\tau/2RC}}{1 + {\rm e}^{-\tau/2RC}}$$
(72)

$$= \frac{V_{\rm s}(1+{\rm e}^{-\tau/2RC})}{1+{\rm e}^{-\tau/2RC}} - \frac{2V_{\rm s}{\rm e}^{-\tau/2RC}}{1+{\rm e}^{-\tau/2RC}}$$
(73)

$$= V_{\rm s} \frac{1 - {\rm e}^{-\tau/2RC}}{1 + {\rm e}^{-\tau/2RC}} \tag{74}$$

also approaches zero, so this case does not provide an energy-efficient way to switch a load across a fixed desired voltage range V, because this will require a large source voltage swing $V_{\rm s}$ in order to compensate for the large damping factor. Solving (75) for the $V_{\rm s}$ required to yield a given desired target load swing of V gives

$$V_{\rm s} = V \frac{1 + e^{-\tau/2RC}}{1 - e^{-\tau/2RC}},\tag{75}$$

which for $\tau \to 0$ approaches (using the general fact that $e^{-x} \to 1 - x$ as $x \to 0$)

$$V_{\rm s} \to \left(\frac{4RC}{\tau} - 1\right) V$$
 (76)

which approaches ∞ in proportion to the frequency f. Also in terms of V, the energy dissipation per half-cycle is (combining eqs. 67, 69, and 75)

$$E_{\rm d,tr} = \frac{1}{2} C V^2 \frac{1 - e^{-\tau/RC}}{(1 - e^{-\tau/2RC})^2}.$$
(77)

Although it is perhaps not immediately obvious, this can be simplified a bit, since in general $1 - e^{-x}$ can be factored into $(1 + e^{-x/2})(1 - e^{-x/2})$, so that

$$E_{\rm d,tr} = \frac{1}{2} C V^2 \frac{(1 + e^{-\tau/2RC})(1 - e^{-\tau/2RC})}{(1 - e^{-\tau/2RC})^2}$$
(78)

$$= \frac{1}{2}CV^{2}\frac{1+\mathrm{e}^{-\tau/2RC}}{1-\mathrm{e}^{-\tau/2RC}},$$
(79)

which clearly (since the numerator is always greater than the denominator) always exceeds the $CV^2/2$ dissipation that would be required for the load to swing through an ordinary step up in the source voltage by V if more time were available. In figure 6 we plot eq. 79 divided by $CV^2/2$ as a function of the relative frequency r = RCf. As $\tau \to 0$, eq. 79 approaches

$$E_{\rm d,tr} \rightarrow \frac{1}{2}CV^2 \frac{1 + (1 - \tau/2RC)}{1 - (1 - \tau/2RC)}$$
 (80)

$$= \frac{1}{2}CV^2 \frac{2-\tau/2RC}{\tau/2RC} \tag{81}$$

$$= \frac{1}{2}CV^2\left(\frac{4RC}{\tau} - 1\right) \tag{82}$$

$$\rightarrow 2CV^2 \frac{RC}{\tau},$$
 (83)

which tends towards infinity in proportion to f. So, although overdriving a load via a large-voltage-swing source can cause the load to traverse a desired voltage swing V in a time $\tau/2$ that may be less than the usual full-swing delay of several RC, this approach is less energy efficient than even ordinary square wave charging driven by the desired voltage swing over clock periods of many RC.

More explicitly, using the definition (33) and eqs. 208 and 79, the energy transfer efficiency for the square wave driver is

$$\eta_{\rm E,tfr} = \frac{E_{\rm tfr}}{E_{\rm d,tr}} \tag{84}$$

$$= \frac{1 - e^{-\tau/2RC}}{1 + e^{-\tau/2RC}},$$
(85)



Figure 6: Energy dissipated per load transition as a multiple of energy transferred (reciprocal of $\eta_{\rm E,tfr}$) for a square wave driver having a frequency-dependent amplitude that is sufficient to swing a load through a fixed voltage range of V.

which approaches 0 for $\tau \to 0$ $(f \to \infty)$ and 1 for $\tau \to \infty$ $(f \to 0)$.

Let's now look at the energy supplied and recovered, and the energy recovery efficiency. Recall that our convention is to measure these relative to $V_{\rm s,min}$. For the square wave, all of the energy is supplied during the first half-cycle, when $v(t) - V_{\rm s,min} = V_{\rm s}$; during the second half-cycle, when $v(t) - V_{\rm s,min} = 0$, no energy is considered to be transmitted or received by the source, regardless of the value of the instantaneous current *i*. Thus, the energy supplied by the source (as defined by eq. 10) is given by

$$E_{\rm sup} = V_{\rm s} \int_{t=0}^{\tau/2} i(t) \,\mathrm{d}t$$
 (86)

$$= V_{\rm s}Q_{\rm tfr} = V_{\rm s}CV \tag{87}$$

$$= CV^2 \frac{1 + e^{-\tau/2RC}}{1 - e^{-\tau/2RC}}$$
(88)

in terms of the load voltage swing V (where in line 88 we substituted eq. 75 for $V_{\rm s}$), while by doubling (79) we get that the energy dissipated per cycle is

$$E_{\rm d,cyc} = CV^2 \frac{1 + e^{-\tau/2RC}}{1 - e^{-\tau/2RC}}.$$
(89)

Since (88) and (89) are the same, the energy recovered by the source is, by the definition (28), always

$$E_{\rm rec} = E_{\rm sup} - E_{\rm d,cyc} = 0 \tag{90}$$

for the square wave. The Q factor for driving a load with a square wave is therefore 1, the lowest possible.

To make sure everything is self-consistent, let's now double-check this result carefully using a different equation (30) for $E_{\rm rec}$ instead:

$$E_{\rm rec} = E_{\rm tfr} + E_{\rm gap} - E_{\rm d,tr} \tag{91}$$

$$= \frac{1}{2}CV^{2} + Q_{\rm tfr}V_{\rm gap} - \frac{1}{2}CV^{2}\frac{1 + e^{-\tau/2RC}}{1 - e^{-\tau/2RC}}$$
(92)

$$= \frac{1}{2}CV^{2}\left(1 - \frac{1 + e^{-\tau/2RC}}{1 - e^{-\tau/2RC}}\right) + Q_{\rm tfr}V_{\rm gap}$$
(93)

$$= \frac{1}{2}CV^{2}\left(\frac{1-\mathrm{e}^{-\tau/2RC}}{1-\mathrm{e}^{-\tau/2RC}} - \frac{1+\mathrm{e}^{-\tau/2RC}}{1-\mathrm{e}^{-\tau/2RC}}\right) + CVv_{0}$$
(94)

$$= -\frac{1}{2}CV^{2}\frac{2\mathrm{e}^{-\tau/2RC}}{1-\mathrm{e}^{-\tau/2RC}} + CV^{2}\frac{\mathrm{e}^{-\tau/2RC}}{1-\mathrm{e}^{-\tau/2RC}}$$
(95)

$$= -CV^{2} \frac{\mathrm{e}^{-\tau/2RC}}{1 - \mathrm{e}^{-\tau/2RC}} + CV^{2} \frac{\mathrm{e}^{-\tau/2RC}}{1 - \mathrm{e}^{-\tau/2RC}}$$
(96)

$$= 0.$$
 (97)

Thus, even though for small τ the energy $E_{\rm tfr}$ transferred to the load is small compared to the energy $E_{\rm d,tr}$ dissipated during charging, the gap energy $E_{\rm gap}$

makes up for the difference; together, the energy transferred plus the gap energy supplies exactly the energy that will be dissipated on the discharging transition, when the source voltage is 0 relative to $V_{\rm s,min}$.

To understand in yet another way why the energy recovery efficiency remains 0 in the high-frequency limit, when the amount of energy usefully transferred onto the load is much less than the energy supplied, consider the following. In the limit $f \to \infty$, the load voltage v(t) is essentially a constant $v(t) \approx$ $V_{\rm avg}$, because there is simply insufficient time for the load voltage to change significantly during the very short charge/discharge cycle. Thus, with the source voltage always at 0 or $V_{\rm s}$, there is always an absolute voltage drop across the resistor of almost exactly a constant $|\Delta V| \approx V_{\rm avg} = V_{\rm s} - V_{\rm avg} = V_{\rm s}/2$. The absolute current is therefore always almost exactly a constant $|I| \approx |\Delta V|/R =$ $V_{\rm s}/2R$, and the power dissipation is almost exactly a constant $P_{\rm d} = I\Delta V =$ $|\Delta V|^2/R = V_{\rm s}^2/4R$. Meanwhile, the instantaneous power $p_{\rm s}$ supplied by the source, relative to the $V_{\rm s,min} = 0$ reference level, is $IV_{\rm s} = V_{\rm s}^2/2R$ exactly half of the time (namely, when $v_{\rm s}(t) = V_{\rm s,max} = V_{\rm s}$) and is 0 the rest of the time (when $v_{\rm s}(t) = V_{\rm s,min} = 0$; so on average, it is $P_{\rm sup} = IV_{\rm s}/2 = V_{\rm s}^2/4R$. Thus, the average power dissipated $P_{\rm d}$ is equal to the power supplied $P_{\rm sup}$; therefore the average power recovered $P_{\rm rec} = P_{\rm sup} - P_{\rm d} = 0$, so the energy recovery efficiency on each cycle is zero also.

To summarize the results of this section, we found that, the energy transfer efficiency of the square-wave driver is 1 at low frequencies, but approaches 0 in inverse proportion to the frequency at high frequencies, meaning that the energy dissipation noticeably worsens if we try to drive the load through a given voltage swing much faster than a characteristic frequency of roughly $f_c = 0.1/RC$, as we can see by looking at figure 6. Meanwhile, the energy recovery efficiency of the square-wave driver is always zero regardless of the frequency. Thus, a squarewave driver cannot perform any energy recovery whatsoever, no matter how low the switching frequency. The square wave therefore represents in a sense the absolute worst case for the energy efficiency among clock-driven dynamic logics; in terms of the achievable energy recovery, logic nodes driven by a square wave are not being driven the least bit adiabatically, no matter how slowly the actual charge transfer takes place due to the nonzero RC time constant. The root cause is that the source voltage toggles between two extremal values in a time that is small compared to RC; this fact by itself immediately and automatically precludes us from achieving any amount of energy recovery.

The square wave having been dispensed with, we now proceed to analyze some more favorable waveform shapes.

7 Linear ramp driver

Suppose now that instead of undergoing instantaneous steps between extremal levels, the source voltage is $v_{\rm s}(t) = V_{\rm s,min} = 0$ for all t < 0 and then begins linearly ramping up at t = 0 with a slope of s; that is, for $t \ge 0$, let

$$v_{\rm s}(t) = st. \tag{98}$$

The differential equation (5) in this case is therefore a linear equation,

$$RC\frac{\mathrm{d}v}{\mathrm{d}t} = st - v,\tag{99}$$

or, rearranged into a certain standard form,

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{1}{RC}v = \frac{s}{RC}t.$$
(100)

Its solution is therefore generated using a standard solution template as

$$v(t) = \frac{\int u(t) \frac{st}{RC} \, \mathrm{d}t + K_1}{u(t)}$$
(101)

for some constant K_1 , where

$$u(t) = \exp\left(\int \frac{1}{RC} \,\mathrm{d}t\right) \tag{102}$$

is the integrating factor. Integrating this, we find that

$$u(t) = \mathrm{e}^{t/RC + K_2} \tag{103}$$

 \mathbf{SO}

$$v(t) = \frac{\int e^{t/RC + K_2} \frac{st}{RC} dt + K_1}{e^{t/RC + K_2}}.$$
 (104)

The integral in the numerator of (104) evaluates as

$$\frac{s}{RC} \int t e^{t/RC + K_2} dt = s(t - RC) e^{t/RC + K_2} + K_3$$
(105)

so letting $K_4 = K_1 + K_3$, and $K_5 = K_4 e^{-K_2}$, (104) simplifies to

$$v(t) = s(t - RC) + K_5 e^{-t/RC}.$$
 (106)

Since we know that v(t) = 0 for t = 0, this implies $K_5 = sRC$, so (106) becomes

$$v(t) = s[t - RC(1 - e^{-t/RC})].$$
(107)

Differentiating, the slope of the load voltage is

$$\frac{\mathrm{d}v}{\mathrm{d}t} = s \left(1 - \mathrm{e}^{-t/RC} \right),\tag{108}$$

and so the instantaneous current is

$$i(t) = C \frac{\mathrm{d}v}{\mathrm{d}t} = sC \left(1 - \mathrm{e}^{-t/RC}\right) \tag{109}$$

and the instantaneous power dissipation is

$$p_{\rm d}(t) = [i(t)]^2 R$$
 (110)

$$= R(sC)^2 \left(1 - e^{-t/RC}\right)^2$$
(111)

$$= C v_{\rm s}^2 \frac{RC}{t^2} \left(1 - 2 {\rm e}^{-t/RC} + {\rm e}^{-2t/RC} \right), \qquad (112)$$

or, in terms of $\epsilon_t = e^{-t/RC}$ and the time-dependent energy coefficient $c_{\rm E}(t) = C v_{\rm s}^2 RC = C s^2 t^2 RC$,

$$p_{\rm d}(t) = \frac{c_{\rm E}}{t^2} (1 - 2\epsilon_t + \epsilon_t^2) \tag{113}$$

$$= Cs^2 RC(1 - 2\epsilon_t + \epsilon_t^2) \tag{114}$$

$$= p_{\infty}(1 - 2\epsilon_t + \epsilon_t^2), \qquad (115)$$

where $p_{\infty} = s^2 R C^2 = p_d(\infty)$ (since $\epsilon_{\infty} = 0$) is the limiting steady-state power dissipation that would be asymptotically approached in the limit $t \to \infty$. (Of course, in practice the source voltage cannot really continue rising forever.)

The cumulative energy dissipation up through time $t = \tau$, when $v_{\rm s}(t) = s\tau$, is therefore

$$E_{\rm d}(t<\tau) = \int_{t=0}^{\tau} p_{\rm d}(t) \,\mathrm{d}t \tag{116}$$

$$= p_{\infty} \int_{t=0}^{\tau} \left(1 - 2e^{-t/RC} + e^{-2t/RC} \right) dt$$
 (117)

$$= p_{\infty} \left[t + 2RC\epsilon_t - \frac{RC}{2}\epsilon_t^2 \right]_{t=0}^{\tau}$$
(118)

$$= p_{\infty} \left\{ \tau - RC \left[2\left(1 - \epsilon_{\tau}\right) - \frac{1}{2}\left(1 - \epsilon_{\tau}^{2}\right) \right] \right\}$$
(119)

$$= E\tau \left[1 - \frac{RC}{\tau} \left(\frac{3}{2} - 2\epsilon_{\tau} + \frac{1}{2}\epsilon_{\tau}^2 \right) \right].$$
 (120)

where we define $E_{\tau} = p_{\infty} \tau$. At time τ , the voltage across the resistor is

$$\Delta V(\tau) = v_{\rm s}(\tau) - v(\tau) = i(t)R = sRC\left(1 - e^{-\tau/RC}\right) = v_{\rm s}\frac{RC}{\tau}(1 - \epsilon_{\tau}).$$
 (121)

If the linear source voltage ramp were to flatten out at time τ and remain flat at the level $V_{\rm s} = v_{\rm s}(t) = s\tau$ for all $t \geq \tau$, then the load voltage eventually goes full-swing, $V = V_{\rm s}$, and the remaining energy dissipated over times $t > \tau$ can be derived using the classic constant-source dissipation for an initial voltage drop of size $\Delta V(\tau)$:

$$E_{\rm d}(t > \tau) = \frac{1}{2} C[\Delta V(\tau)]^2$$
 (122)

$$= \frac{1}{2}C\left[sRC(1-\epsilon_{\tau})\right]^2 \tag{123}$$

$$= \frac{1}{2}CV^2 \left(\frac{RC}{\tau}\right)^2 \left(1-\epsilon_{\tau}\right)^2.$$
(124)

Note also that since $s = V/\tau$, we have that the energy E_{τ} that appears in equation 120 can be written as

$$E_{\tau} = p_{\infty}\tau = Cs^2 R C\tau = C \frac{V^2}{\tau^2} R C\tau = C V^2 \frac{RC}{\tau} = c_{\rm E}/\tau, \qquad (125)$$

where $c_{\rm E}$ is called the *energy coefficient* $c_{\rm E} = CV^2RC$ for the adiabatic charge transfer. Thus, (124) can be written more simply as just

$$E_{\rm d}(t > \tau) = \frac{1}{2} E_{\tau} \frac{RC}{\tau} (1 - \epsilon_{\tau})^2$$
 (126)

$$= \frac{1}{2} \frac{c_{\rm E}}{\tau} (1 - \epsilon_{\tau})^2 \,. \tag{127}$$

Summing equations (120) and (126), the total energy dissipation for a ramp that goes to a voltage $V = V_s$ over time τ and then levels off is given by

$$E_{\mathrm{d,tr}} = E_{\mathrm{d}}(t < \tau) + E_{\mathrm{d}}(t > \tau)$$

$$(128)$$

$$= E_{\tau} \left[1 - \frac{RC}{\tau} \left(\frac{3}{2} - 2\epsilon_{\tau} + \frac{1}{2}\epsilon_{\tau}^2 \right) \right] + \frac{1}{2} E_{\tau} \frac{RC}{\tau} \left(1 - \epsilon_{\tau} \right)^2$$
(129)

$$= E_{\tau} \left\{ 1 + \frac{RC}{\tau} \left[-\left(\frac{3}{2} - 2\epsilon_{\tau} + \frac{1}{2}\epsilon_{\tau}^2\right) + \left(\frac{1}{2} - \epsilon_{\tau} + \frac{1}{2}\epsilon_{\tau}^2\right) \right] \right\} (130)$$

$$= E_{\tau} \left[1 + \frac{RC}{\tau} \left(\epsilon_{\tau} - 1 \right) \right]$$
(131)

$$= CV^{2}r\left[1+r\left(e^{-1/r}-1\right)\right]$$
(132)

where in the last line we have substituted the rapidity $r = RC/\tau$ of the voltage source's transition interval τ relative to the $t_c = RC$ time constant of the circuit. See figure 8 for a graph of this function. As we can see from the graph, expression 132 approaches $CV^2/2$ as the rapidity $r \to \infty$, which is to be expected since in that limit $\tau \to \infty$ and the ramp approaches the instantaneous rise of the step function. To confirm this limit analytically, we'll let $\varepsilon = 1/r = \tau/RC$ be the rise time in units of $t_c = RC$, and utilize the Taylor series expansion,

$$e^{-\varepsilon} = \sum_{n=0}^{\infty} \frac{(-1)^n \varepsilon^n}{n!} = 1 - \varepsilon + \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{6} + \frac{\varepsilon^4}{24} - \dots$$
(133)

Thus we find that

$$\frac{E_{\rm d,tr}}{CV^2} = r \left[1 + r \left(e^{-1/r} - 1 \right) \right]$$
(134)

$$= \frac{1}{\varepsilon} \left[1 + \frac{1}{\varepsilon} \left(e^{-\varepsilon} - 1 \right) \right]$$
(135)

$$= \frac{1}{\varepsilon} \left\{ 1 + \frac{1}{\varepsilon} \left[\left(1 - \varepsilon + \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{6} + \frac{\varepsilon^4}{24} - \dots \right) - 1 \right] \right\}$$
(136)

$$= \frac{1}{\varepsilon} \left[1 + \frac{1}{\varepsilon} \left(-\varepsilon + \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{6} + \frac{\varepsilon^4}{24} - \dots \right) \right]$$
(137)



Figure 7: Energy dissipated per load transition as a multiple of energy transferred (reciprocal of $\eta_{\rm E,tfr}$) as a function of the rapidity factor $r = RC/\tau$ for a linear ramp driver with fixed source voltage swing $V = V_{\rm s}$ and rise time τ .

$$= \frac{1}{\varepsilon} \left[1 + \left(-1 + \frac{\varepsilon}{2} - \frac{\varepsilon^2}{6} + \frac{\varepsilon^3}{24} - \dots \right) \right]$$
(138)

$$= \frac{1}{\varepsilon} \left(\frac{\varepsilon}{2} - \frac{\varepsilon^2}{6} + \frac{\varepsilon^3}{24} - \dots \right)$$
(139)

$$= \frac{1}{2} - \frac{\varepsilon}{6} + \frac{\varepsilon^2}{24} - \dots$$
 (140)

So, as the rise time τ goes to zero, we have that $E_{\rm d,tr}$ goes to

$$\lim_{\varepsilon \to 0} E_{\mathrm{d,tr}} = \frac{1}{2} C V^2 \tag{141}$$

as expected. The other limit of (132) can be found even more easily by observing that as $r \to 0 \ (\tau \to \infty)$,

$$\frac{E_{\rm d,tr}}{CV^2} = r \left[1 + r \left(e^{-1/r} - 1 \right) \right]$$
(142)

$$\rightarrow r \left[1 + r \left(e^{-\infty} - 1\right)\right] \tag{143}$$

$$= r [1 + r (0 - 1)]$$
(144)
= $r - r^{2}$ (145)

$$\rightarrow r$$
 (146)

$$\rightarrow 0.$$
 (147)

Thus, the energy dissipated approaches 0 as the rise time approaches infinity, as expected for an adiabatic charge transfer.

The exact result (132) for a real voltage ramp may be contrasted with the more idealized situation where one imagines that there is an exactly constant current of $i = CV/\tau$ (provided by some unspecified current source) passing through the resistor for a time τ , in which case the power is a constant $CV^2RC/\tau^2 = c_{\rm E}/\tau^2$ and the energy dissipated is exactly $E_{\tau} = c_{\rm E}/\tau$. Equation (132) is not precisely equal to this ideal because the load does not exactly follow the ramp; the slope of the load voltage initially lags, and then later gradually levels off after the source voltage has flattened. These variations from the linear-ramp ideal result in the additional correction terms appearing in eq. 132.

Note, however, that for small values of the rapidity $r = RC/\tau \ll 1$, that is when the charging time $\tau \gg RC$, the value of $\epsilon_{\tau} = e^{-1/r}$ converges exponentially rapidly towards zero as $r \to 0$, and so very rapidly $r \left[1 + r \left(e^{-1/r} - 1\right)\right] \to$ r(1-r) which approaches r for small r, so that in this regime (132) to first order approaches simply

$$E_{\rm d} \to C V^2 \frac{RC}{\tau} = \frac{c_{\rm E}}{\tau},$$
 (148)

which is the classic formula for adiabatic charging with constant current over time τ .

The energy transfer efficiency of the ideal adiabatic charge transfer would be

$$\eta_{\rm E,tfr}({\rm ideal}) = \frac{CV^2/2}{CV^2 \frac{RC}{\tau}} = \frac{\tau}{2RC} = \frac{1}{2r},$$
(149)

thus decreasing in inverse proportion to the rapidity of the voltage rise; as the rapidity $r \to 0$, we have $\eta_{\text{E,tfr}} \to \eta_{\text{E,tfr}}(\text{ideal}) = \frac{\tau}{2RC} \to \infty$. Thus the energy transfer efficiency of the voltage ramp approaches infinity in proportion to the rise time τ of the source, which is the expected slow-speed behavior for an adiabatic charge transfer.

This idealized formula (149) would also seem to suggest that the energy transfer efficiency would become indefinitely small as the rapidity increases; however by examining the exact dissipation (132) and substituting (140) we can see that the exact energy transfer efficiency is

$$\eta_{\rm E,tfr} = \frac{CV^2/2}{CV^2 r \left[1 + r \left(e^{-1/r} - 1\right)\right]}$$
(150)

$$= \frac{1}{2r\left[1+r\left(e^{-1/r}-1\right)\right]}$$
(151)

$$= \frac{1}{2\left(\frac{1}{2} - \frac{\varepsilon}{6} + \frac{\varepsilon^2}{24} - \ldots\right)} \tag{152}$$

$$= \frac{1}{1 - \frac{\varepsilon}{3} + \frac{\varepsilon^2}{12} - \dots},$$
 (153)

which approaches 1 (from above) as $\varepsilon \to 0$; thus we can see that the energy transfer efficiency is in reality at worst 1. This also follows even more simply from eq. 141.

As for the energy recovery efficiency $\eta_{\rm E,rec}$, this is equally easy to compute. Since $V = V_{\rm s}$ in this case, $V_{\rm gap} = 0$ and so $E_{\rm gap} = 0$. Thus, $E_{\rm sup} = E_{\rm tfr} + E_{\rm d,tr}$ and $E_{\rm rec} = E_{\rm tfr} - E_{\rm d,tr}$. So the energy recovery efficiency is just

$$\eta_{\rm E,rec} = \frac{E_{\rm rec}}{E_{\rm sup}} \tag{154}$$

$$= \frac{E_{\rm tfr} - E_{\rm d,tr}}{E_{\rm tfr} + E_{\rm d,tr}}$$
(155)

$$= \frac{\frac{1}{2}CV^2 - CV^2r\left[1 + r\left(e^{-1/r} - 1\right)\right]}{\frac{1}{2}CV^2 + CV^2r\left[1 + r\left(e^{-1/r} - 1\right)\right]}$$
(156)

$$= \frac{\frac{1}{2}CV^2 - CV^2\left(\frac{1}{2} - \frac{\varepsilon}{6} + \frac{\varepsilon^2}{24} - \dots\right)}{\frac{1}{2}CV^2 + CV^2\left(\frac{1}{2} - \frac{\varepsilon}{6} + \frac{\varepsilon^2}{24} - \dots\right)}$$
(157)

$$= \frac{\frac{1}{2} - \left(\frac{1}{2} - \frac{\varepsilon}{6} + \frac{\varepsilon}{24} - \dots\right)}{\frac{1}{2} + \left(\frac{1}{2} - \frac{\varepsilon}{6} + \frac{\varepsilon^2}{24} - \dots\right)}$$
(158)

$$= \frac{\frac{\varepsilon}{6} - \frac{\varepsilon^2}{24} + \dots}{1 - \frac{\varepsilon}{6} + \frac{\varepsilon^2}{24} - \dots}.$$
(159)

As $\varepsilon \to 0$, this expression approaches $\varepsilon/6 = \tau/6RC$, so we see that for short rise times, the fraction of energy that's recovered is the same as the rise time expressed as a fraction of $6t_c = 6RC$, or six RC time constants.



Figure 8: Energy recovery efficiency $\eta_{\rm E,rec}$ as a function of the rapidity factor $r = RC/\tau$ for a linear ramp driver with fixed source and load voltage swing $V = V_{\rm s}$ and rise time τ .

Of course, in the slow charging limit $\tau \to \infty$ (ε), we saw earlier that $E_{\rm d,tr} \to 0$, so in that limit

$$\eta_{\rm E,rec} = \frac{E_{\rm tfr} - E_{\rm d,tr}}{E_{\rm tfr} + E_{\rm d,tr}}$$
(160)

$$\rightarrow \quad \frac{E_{\rm tfr} - 0}{E_{\rm tfr} + 0} \tag{161}$$

$$= 1,$$
 (162)

and thus the energy recovery efficiency approaches 100% in this case.

8 Trapezoidal waveforms

Figure 9 shows example source and load waveforms for a trapezoidal source waveform. The source wave shape is characterized by two parameters: $\tau_{\rm s,tr}$ is the rise (or fall) time of the source waveform, which $\tau_{\rm cyc} = 1/f_{\rm cyc}$ is the total cycle period for a wave with cycle

[This section still needs to be written.]



Figure 9: Source and load waveforms for a trapezoidal-wave driver with clock period $\tau_{\rm cyc} \approx 5.26 RC$ and rise time $\tau_{\rm s,tr} = \tau_{\rm cyc}/4 = \approx 1.32 RC$. The time and voltage intervals are scale-free, in arbitrary units.

9 Sinusoidal drivers

[This section still needs to be cleaned up.]

In this section, for greater simplicity of the formulas we assume a voltage reference based on the average signal level $V_{\text{avg}} = 0$. Consider now an ideal sinusoidal voltage source

$$v_{\rm s} = V_{\rm avg} + V_{\rm sa}\sin(\omega t) = V_{\rm sa}\sin(\omega t),\tag{163}$$

where $V_{\rm sa}$ is the source voltage amplitude and $\omega = 2\pi f$ is the angular frequency of the source.

Plugging this into our general differential equation (5), we have that

$$RC\frac{\mathrm{d}v}{\mathrm{d}t} = V_{\mathrm{sa}}\sin\omega t - v. \tag{164}$$

The solution to eq. 164 (derived in appendix B) is

$$v(t) = \frac{V_{\rm sa}}{\sqrt{(RC\omega)^2 + 1}} \sin[\omega t - \tan^{-1}(RC\omega)], \qquad (165)$$

where notice that the signal at the load has been taken down in amplitude by the damping factor $d = \sqrt{(RC\omega)^2 + 1} > 1$ and lags in phase by $\theta = \tan^{-1}(RC\omega)$. Both terms depend on the critical dimensionless speed parameter $\sigma = RC\omega = t_c/t_r$ where $t_c = RC$ is the time constant (the e-folding time for the exponential decay) for charging the load C through resistance R, while $t_r = t_{cyc}/2\pi$ is the time for the source signal to rotate 1 radian, where $t_{cyc} = 1/f = 2\pi/\omega$ is the clock cycle period. We might call σ the "quickness" of the clock oscillation, judged relative to the circuit's natural transition time of t_c .

[Continue fixing voltage notation below.]

Now, plugging (165) back into (4), the voltage drop across the resistor is

$$v_{\rm s} - v = \frac{V_{\rm sa}\sigma}{\sqrt{\sigma^2 + 1}}\cos(\omega t - \tan^{-1}\sigma),\tag{166}$$

so by (3) the current is

$$i = \frac{CV_{\rm sa}\omega}{\sqrt{\sigma^2 + 1}}\cos(\omega t - \tan^{-1}\sigma).$$
(167)

Using p = iv, the instantaneous power dissipated in the resistor is then

$$p = \frac{CV_{\rm sa}^2 R C \omega^2}{\sigma^2 + 1} \cos^2(\omega t - \tan^{-1}\sigma).$$
(168)

Over one complete cycle of length $t_{\rm cyc} = 2\pi/\omega$, the energy dissipated is thus

$$E_{\rm d,cyc} = \int_{t=0}^{2\pi/\omega} p \,\mathrm{d}t,$$
 (169)

$$= \frac{CV_{\rm sa}^2 R C \omega^2}{\sigma^2 + 1} \int_{t=0}^{2\pi/\omega} \cos^2(\omega t - \tan^{-1}\sigma) \,\mathrm{d}t$$
 (170)

$$= \frac{CV_{\rm sa}^2 R C \omega^2}{\sigma^2 + 1} \int_{\theta=0}^{2\pi} \cos^2 \theta \, \frac{\mathrm{d}\theta}{\omega} \tag{171}$$

$$= CV_{\rm sa}^2 \frac{\pi\sigma}{\sigma^2 + 1} \tag{172}$$

$$= \frac{\pi}{\sigma + \sigma^{-1}} C V_{\rm sa}^2 \tag{173}$$

where in (171) we have temporarily substituted $\theta = \omega t$ and removed the phase lag, which is irrelevant to the full-cycle integration; the integral then evaluates to π/ω since by the symmetry between sin and cos it is exactly half of

=

$$\int_{0}^{2\pi} (\cos^2\theta + \sin^2\theta) \frac{\mathrm{d}\theta}{\omega} = \int_{0}^{2\pi} 1 \frac{\mathrm{d}\theta}{\omega} = \frac{2\pi}{\omega}.$$
 (174)

Now, the most important thing for us to note about eq. 173 is its behavior for small quickness $\sigma \to 0$, that is for slow charging, when the radial time $t_{\rm r} = t_{\rm cyc}/2\pi \gg RC$. Just as with the classical case of adiabatic charging with a linear ramp, note that here too, as the signal rise time increases and the clock frequency decreases, the energy dissipated per cycle decreases roughly proportionately, since $\sigma/(\sigma^2 + 1) \to \sigma$ as $\sigma \to 0$.

Note also that for very large quickness $\sigma \to \infty$, it is also the case that $E_{d,cyc} \to 0$, since $\sigma/(\sigma^2 + 1) \to 1/\sigma$ as $\sigma \to \infty$. However, in this case, the low dissipation can be attributed to the fact that the load voltage does not have time to change very much in a cycle, due to the substantial size of the damping factor $d = \sqrt{\sigma^2 + 1}$, which approaches σ as $\sigma \to \infty$.

The maximum dissipation per cycle is $E_{\rm d,cyc} = \frac{\pi}{2}CV_{\rm sa}^2$ which occurs when $\sigma = 1$, that is when $\omega = 1/RC$. This is the case of "least adiabatic" charging, but in fact dissipates only $\pi/8 \approx 40\%$ as much energy as the $4CV_{\rm sa}^2$ that would be dissipated by a square wave taking the load through the identical range of voltages $[-V_{\rm sa}, +V_{\rm sa}]$.

In general, we can characterize the *degree of adiabaticity* of a given process as the ratio between the energy transferred $E_{\rm tfr}$ and the energy dissipated $E_{\rm diss}$. For the charging and discharging of a load between $-V_{\rm sa}$ and $+V_{\rm sa}$, the total amount of electrostatic energy moved onto and off of the load is $E_{\rm tfr} = \frac{1}{2}C(2V_{\rm sa})^2 = 2CV_{\rm sa}^2$, whereas with a sinusoidal driver we saw that the actual dissipation in a cycle was only $E_{\rm d,cyc} = CV_{\rm sa}^2\pi\sigma/(\sigma^2+1)$. Therefore, the degree of adiabaticity A of the complete sinusoidal charge/discharge process is

$$A = \frac{2(\sigma^2 + 1)}{\pi\sigma} = \frac{2}{\pi}(\sigma + \sigma^{-1}), \qquad (175)$$

which has a minimum of $4/\pi = 1.27$ when $\sigma = 1$. Or, putting things another way, we can define the *energy recovery efficiency* [fix] $\eta_{\rm E,rec} = 1 - 1/A = (E_{\rm tfr} - E_{\rm diss})/E_{\rm tfr} = E_{\rm rec}$ which is the ratio between the amount of energy recovered (*i.e.*, not dissipated) $E_{\rm rec} = E_{\rm tfr} - E_{\rm diss}$ and the amount of energy transferred. Phrased this way, the efficiency of the sinusoidal charge/discharge cycle is

$$\eta = 1 - \frac{\pi}{2(\sigma + \sigma^{-1})}$$
(176)

whose minimum is

$$\eta = 1 - \pi/4 \approx 21.46\% \tag{177}$$

when $\sigma = 1$, whereas the efficiency approaches 100% as $\sigma \to 0$, with the distance from 100% in that limit being proportional to σ since the expression (176) for η approaches $1 - \pi \sigma/2$. Note, in contrast, that the energy efficiency of a standard abrupt (square wave) charge/discharge process always approaches 0% whenever the load voltage range approaches full-swing, since the energy delivered from the constant-voltage source after the sharp rising clock edge is CV^2 , and exactly this much energy is dissipated upon charging and then discharging the load (half of it or $\frac{1}{2}CV^2$ after each clock edge). Whereas for the adiabatic sinusoidal driver, the case $\sigma \to 0, d \to 1$ where the load voltage range approaches full swing (and also with phase lag θ approaching zero) is also the same limit in which the energy efficiency of the charge transfer approaches 100%.

We can thus see that in all cases, sinusoidal charging dissipates less energy per complete charge-discharge cycle than sharp-edged square-wave charging, dissipating at most about 21% of the energy transferred, and at best nearly 0% when the clock period $t_{\rm cyc}$ is large compared to $2\pi RC$, since in this limit, as $\sigma = RC/t_{\rm cyc} \rightarrow 0$, the fraction of the capacitor charging energy that is actually dissipated on each cycle approaches $\pi\sigma/2$, that is, it goes down in proportion to the quickness of the clock transitions, as would be expected for an asymptotically adiabatic process.

10 Conclusion

[Summarize results and suggest directions for future work, including more accurate modeling of particular resonator architectures, logic devices, and distributed loads.]

Appendix A: Energy transferred and the need for the gap energy

This appendix gives detailed derivations showing why, with our definitions, the energy $E_{\rm tfr}$ transferred onto the load is always $CV^2/2$, and why the gap energy $E_{\rm gap}$ arises in the relation between the energy supplied and energy transferred.

Although it may not be immediately apparent from the defining expression (22), the energy transferred $E_{\rm tfr}$ is actually a state function that depends only on the load capacitance C and the total load voltage swing V, and not on the detailed current/voltage trajectories v(t), i(t) at all. To see this, we need

merely expand the charge increment $\mathrm{d}q$ in terms of the capacitance using (1), and simplify:

$$E_{\rm tfr} = \int_{\substack{t=\tau_0\\ c\tau_1}}^{\tau_1} \mathrm{d}e_{\rm tfr}(t) \tag{178}$$

$$= \int_{t=\tau_0}^{\tau_1} \left[v(t) - v(\tau_0) \right] \, \mathrm{d}q(t) \tag{179}$$

$$= \int_{t=\tau_0}^{\tau_1} \left[v(t) - v(\tau_0) \right] C \, \mathrm{d}v(t)$$
(180)

$$= C \left[\int_{t=\tau_0}^{\tau_1} v(t) \, \mathrm{d}v(t) \right] - C v(\tau_0) \int_{t=\tau_0}^{\tau_1} \, \mathrm{d}v(t)$$
(181)

$$= C \left\{ \frac{1}{2} \left[v(t) \right]^2 \right\}_{t=\tau_0}^{\tau_1} - C v(\tau_0) \left[v(t) \right]_{t=\tau_0}^{\tau_1}$$
(182)

$$= \frac{1}{2}C\left\{\left[v(\tau_0)\right]^2 - \left[v(\tau_1)\right]^2\right\} - Cv(\tau_0)\left[v(\tau_1) - v(\tau_0)\right]$$
(183)

$$= \frac{1}{2}C[v(\tau_1)]^2 - \frac{1}{2}C[v(\tau_0)]^2 - Cv(\tau_0)v(\tau_1) + C[v(\tau_0)]^2 \quad (184)$$

$$= \frac{1}{2}C[v(\tau_1)]^2 - Cv(\tau_0)v(\tau_1) + \frac{1}{2}C[v(\tau_0)]^2$$
(185)

$$= \frac{1}{2}C\left\{\left[v(\tau_1)\right]^2 - 2Cv(\tau_0)v(\tau_1) + \left[v(\tau_0)\right]^2\right\}$$
(186)

$$= \frac{1}{2}C[v(\tau_1) - v(\tau_0)]^2$$
(187)

$$= \frac{1}{2}CV^2. \tag{188}$$

Another, slightly less direct derivation involves integrating over the charge transferred to the load, rather than the voltage:

$$E_{\rm tfr} = \int_{t=\tau_0}^{\tau_1} \mathrm{d}e_{\rm tfr}(t) \tag{189}$$

$$= \int_{t=\tau_0}^{\tau_1} [v(t) - v(\tau_0)] \, \mathrm{d}q(t)$$
(190)

$$= \int_{t=\tau_0}^{\tau_1} \left[\int_{t'=\tau_0}^t dv(t') \right] dq(t)$$
(191)

$$= \int_{t=\tau_0}^{\tau_1} \left[\int_{t'=\tau_0}^t \frac{\mathrm{d}q(t')}{C} \right] \mathrm{d}q(t) \tag{192}$$

$$= \frac{1}{C} \int_{t=\tau_0}^{\tau_1} \left[\int_{t'=\tau_0}^{t} \mathrm{d}q(t') \right] \mathrm{d}q(t)$$
(193)

$$= \frac{1}{C} \int_{t=\tau_0}^{\tau_1} \left[q(t') \right]_{t'=\tau_0}^t \, \mathrm{d}q(t) \tag{194}$$

$$= \frac{1}{C} \int_{t=\tau_0}^{\tau_1} \left[q(t) - q(\tau_0) \right] \, \mathrm{d}q(t) \tag{195}$$

$$= \frac{1}{C} \left\{ \left[\int_{t=\tau_0}^{\tau_1} q(t) \, \mathrm{d}q(t) \right] - \int_{t=\tau_0}^{\tau_1} q(\tau_0) \, \mathrm{d}q(t) \right\}$$
(196)

$$= \frac{1}{C} \left\{ \left[\frac{1}{2} [q(t)]^2 \right]_{t=\tau_0}^{\tau_1} - q(\tau_0) \left[q(t) \right]_{t=\tau_0}^{\tau_1} \right\}$$
(197)

$$= \frac{1}{C} \left\{ \frac{1}{2} \left[[q(\tau_1)]^2 - [q(\tau_0)]^2 \right] - q(\tau_0) \left[q(\tau_1) - q(\tau_0) \right] \right\}$$
(198)

$$= \frac{1}{2C} \left\{ [q(\tau_1)]^2 - [q(\tau_0)]^2 - 2q(\tau_0)q(\tau_1) + 2[q(\tau_0)]^2 \right\}$$
(199)

$$= \frac{1}{2C} \left\{ [q(\tau_1)]^2 - 2q(\tau_0)q(\tau_1) + [q(\tau_0)]^2 \right\}$$
(200)

$$= \frac{1}{2C} \left[q(\tau_1) - q(\tau_0) \right]^2 \tag{201}$$

$$= \frac{1}{2C}Q_{\rm tfr}^2 \tag{202}$$

$$= \frac{1}{2C} (CV)^2$$
(203)

$$= \frac{1}{2}CV^2. \tag{204}$$

In a rather more abbreviated version of this derivation, we simplify things by redefining q to be the instantaneous quantity of charge accumulated on the capacitor relative to whatever charge it was storing at the initial time τ_0 , when $v(\tau_0) = V_{\min}$, so that we have $q(\tau_0) = 0$, and by then changing integration variables to integrate directly over the accumulated charge q rather than the time t, so that for any t we define v(q(t)) = v(t):

$$E_{\rm tfr} = \int \mathrm{d}E_{\rm tfr} \tag{205}$$

$$= \int_{q=0}^{Q \text{ thr}} [v(q) - v(0)] \, \mathrm{d}q$$
 (206)

$$= \int_{q=0}^{Q_{\rm tfr}} \frac{q}{C} \,\mathrm{d}q \tag{207}$$

$$= \frac{Q_{\rm tfr}^2}{2C} = \frac{1}{2}CV^2,$$
 (208)

where we have made use of the fact that, in this new setting, v(q) - v(0) = q/C, where both voltages here are functions of the accumulated charge.

To see more clearly why the gap energy E_{gap} arises, suppose that instead of defining the incremental energy transferred de_{tfr} based on the voltage reference V_{\min} , we had instead used some other voltage reference, such as $V_{\text{s,min}}$. Then the energy transferred onto the capacitance in the process of charging it from voltage V_{\min} to voltage V_{\max} would be, under this alternate definition,

$$E'_{\rm tfr} = e_{\rm tfr}(\tau_1) - e_{\rm tfr}(\tau_0)$$
(209)

$$= \frac{1}{2}C(V_{\rm max} - V_{\rm s,min})^2 - \frac{1}{2}C(V_{\rm min} - V_{\rm s,min})^2$$
(210)

$$= \frac{1}{2}C\left[\left(V_{\max}^2 - 2V_{\max}V_{s,\min} + V_{s,\min}^2\right) - \left(V_{\min}^2 - 2V_{\min}V_{s,\min} + V_{s,\min}^2\right)\right]\right)$$

$$= \frac{1}{2}C\left[V_{\max}^2 - 2V_{\max}V_{s,\min} - V_{\min}^2 + 2V_{\min}V_{s,\min}\right]$$
(212)

$$= \frac{1}{2} C \left[V_{\max}^2 - 2V_{s,\min} (V_{\max} - V_{\min}) - V_{\min}^2 \right]$$
(213)

$$= \frac{1}{2} C \left[V_{\max}^2 - 2V_{s,\min} V - V_{\min}^2 \right].$$
 (214)

The difference between this and our original definition of $E_{\rm tfr}$ would be:

$$E'_{\rm tfr} - E_{\rm tfr} = \frac{1}{2} C \left[V_{\rm max}^2 - 2V_{\rm s,min} V - V_{\rm min}^2 \right] - \frac{1}{2} C V^2$$
(215)

$$= \frac{1}{2}C\left[V_{\max}^2 - 2V_{s,\min}V - V_{\min}^2 - (V_{\max} - V_{\min})^2\right]$$
(216)

$$= \frac{1}{2}C\left[V_{\max}^2 - 2V_{s,\min}V - V_{\min}^2 - (V_{\max}^2 - 2V_{\max}V_{\min} + V_{\min}^2)\right]$$

$$= \frac{1}{2}C\left[-2V_{\rm s,min}V + 2V_{\rm max}V_{\rm min} - 2V_{\rm min}^2\right]$$
(218)

$$= C \left[-V_{\rm s,min} V + V_{\rm max} V_{\rm min} - V_{\rm min}^2 \right]$$
(219)

$$= C \left[-V_{\text{s,min}} V_{\text{max}} + V_{\text{s,min}} V_{\text{min}} + V_{\text{max}} V_{\text{min}} - V_{\text{min}}^2 \right]$$
(220)

$$= C [V_{\max}(V_{\min} - V_{s,\min}) - V_{\min}(V_{\min} - V_{s,\min})]$$
(221)

$$= C \left(V_{\max} V_{gap} - V_{\min} V_{gap} \right)$$

$$= C \left(V_{\max} V_{gap} - V_{\min} V_{gap} \right)$$
(222)
(222)

$$- CVV$$

$$(224)$$

$$= Q_{tfr}V_{gap}$$
(224)
$$= Q_{tfr}V_{gap}$$
(225)

$$= E_{\text{gap}}.$$
 (226)

Appendix B: Solution of differential equation from §9

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This appendix (which may or may not be useful to include in a published paper) shows how to solve the differential equation (164) from first principles, without delving into formulations in terms of complex impedances which may be non-intuitive for some readers.

We know from our general background knowledge that a circuit composed of linear elements (such as resistors, capacitors, and inductors) and driven by constant-frequency sinusoidal sources will always attain what is known as an AC steady state. Thus, the solution to (164) must also be a sinusoid of constant amplitude, frequency, and phase shift; furthermore, it must have the same average (DC) voltage level as the source, since with no net DC current the resistor cannot maintain a constant DC voltage drop across it. Thus, we know that the solution to (164) must be of the general form

$$v(t) = V_{\rm a} \sin(\omega_{\rm L} t + \theta_{\rm L}) \tag{227}$$

where $V_{\rm a}$ is the amplitude of the voltage swing (from $-V_{\rm a}$ to $+V_{\rm a}$) of the signal on the load node, $\omega_{\rm L}$ is the angular frequency of this signal (which we will see must be the same as the driving frequency ω), and $\theta_{\rm L}$ is the phase of the load voltage relative to the driving voltage at t = 0 (and also at all other times, since $\omega_{\rm L} = \omega$).

Starting from (227), we can take its derivative

$$\frac{\mathrm{d}v}{\mathrm{d}t} = V_{\mathrm{a}}\omega_{\mathrm{L}}\cos(\omega_{\mathrm{L}}t + \theta_{\mathrm{L}}) \tag{228}$$

which we can then plug into the left side of (164), along with (227) itself in place of v on the right, to get:

$$V_{\rm a}\omega_{\rm L}\cos(\omega_{\rm L}t + \theta_{\rm L}) = \frac{V_{\rm sa}\sin\omega t - V_{\rm a}\sin(\omega_{\rm L} + \theta_{\rm L})}{RC}.$$
 (229)

We now have an ordinary (no longer differential) equation which we need merely solve in order to find the unknown parameters $V_{\rm a}$, $\omega_{\rm L}$, and $\theta_{\rm L}$ as functions of the known parameters $V_{\rm sa}$, ω , and RC. Since the equation must hold true for all values of $t \in (-\infty, +\infty)$, this will allow us to determine all three unknown parameters using just this single equation.

In what follows, we often substitute $t_c = RC$ for conciseness. Multiplying both sides of (229) by t_c and gathering all of the terms containing unknowns on the left side of the equation, we get

$$t_{\rm c} V_{\rm a} \omega_{\rm L} \cos(\omega_{\rm L} t + \theta_{\rm L}) + V_{\rm a} \sin(\omega_{\rm L} t + \theta_{\rm L}) = V_{\rm sa} \sin \omega t.$$
(230)

Factoring out $V_{\rm a}$ from the left side,

$$V_{\rm a} \left[t_{\rm c} \omega_{\rm L} \cos(\omega_{\rm L} t + \theta_{\rm L}) + \sin(\omega_{\rm L} t + \theta_{\rm L}) \right] = V_{\rm sa} \sin \omega t.$$
(231)

To match this up with the right-hand side, we would prefer if the left-hand side was expressed as a single sinusoidal function of t. Fortunately, the term in brackets is of the form $a \cos x + \sin x$ which reduces to a single sinusoidal function. In other words, we can always write

$$a\cos x + \sin x = b\sin(x+\phi) \tag{232}$$

where b and ϕ are both closed-form functions of a. To see this, note that equation (232) is merely the real part of

$$ae^{i(x+\pi/2)} + e^{ix} = be^{i(x+\phi)}.$$
 (233)

Factoring the exponentials,

$$a\mathrm{e}^{\mathrm{i}x}\mathrm{e}^{\mathrm{i}\pi/2} + \mathrm{e}^{\mathrm{i}x} = b\mathrm{e}^{\mathrm{i}x}\mathrm{e}^{\mathrm{i}\phi},\tag{234}$$

and we can divide out e^{ix} , leaving

$$a\mathbf{i} + 1 = b\mathbf{e}^{\mathbf{i}\phi}.\tag{235}$$

This equation makes it obvious that

$$\phi = \tan^{-1} a, \qquad (236)$$

$$b = \sqrt{a^2 + 1},$$
 (237)

so the desired identity is

$$a\cos x + \sin x = \sqrt{a^2 + 1}\sin(x + \tan^{-1}a).$$
 (238)

This is exactly the sort of thing we need to simplify the left-hand side of equation (231). The square root and arctangent functions do not present a problem since those expressions are just constants (not functions of x, or in our case t). Applying (238) to (231), we get

$$V_{\rm a}\sqrt{(t_{\rm c}\omega_{\rm L})^2 + 1}\sin\left[\omega_{\rm L}t + \theta_{\rm L} + \tan^{-1}(t_{\rm c}\omega_{\rm L})\right] = V_{\rm sa}\sin\omega t.$$
 (239)

Since $t_c \omega_L$ appears twice, we begin using σ in place of it:

$$V_{\rm a}\sqrt{\sigma^2 + 1\sin(\omega_{\rm L}t + \theta_{\rm L} + \tan^{-1}\sigma)} = V_{\rm sa}\sin\omega t.$$
 (240)

Now, an equation between two sinusoidal functions of t can only hold true for all values of t if the frequencies, amplitudes, and phases of these two functions are all identical. Thus from (240) we obtain the three equations:

$$\omega_{\rm L} = \omega \tag{241}$$

$$V_{\rm a}\sqrt{\sigma^2 + 1} = V_{\rm sa}, \qquad (242)$$

$$\theta_{\rm L} + \tan^{-1}\sigma = 0. \tag{243}$$

Solving for $V_{\rm a}$ and $\theta_{\rm L}$, we have

$$V_{\rm a} = \frac{V_{\rm sa}}{\sqrt{\sigma^2 + 1}} \tag{244}$$

$$\theta_{\rm L} = -\tan^{-1}\sigma. \tag{245}$$

At this point, we observe that the voltage swing on the load is taken down by the damping divisor $d = \sqrt{\sigma^2 + 1}$, and that the load incurs a phase lag of $\theta = -\theta_{\rm L} = \tan^{-1} \sigma$. Plugging the equations for $V_{\rm a}$ and $\theta_{\rm L}$ back into (227), we finally have that the unique real solution to the differential equation (164), modulo physically meaningless shifts of $\pm 2\pi n$ in the phase lag θ , is

$$v(t) = \frac{V_{\rm sa}}{\sqrt{\sigma^2 + 1}} \sin(\omega - \tan^{-1}\sigma).$$
(246)