Problems with Equality Constraints

- Realistic optimization problems have constraints.
- If these constraints are equality constraints, then the optimization methods developed in the previous lecture can be modified to solve the problem with constraints.

Problem Formulation

$$minimize\ f(x_1, x_2, ..., x_n, u_1, u_2, ..., u_m) \tag{1}$$

subject to

$$\begin{aligned} g_1(x_1, x_2, ..., x_n, u_1, u_2, ..., u_m) &= 0 \\ g_2(x_1, x_2, ..., x_n, u_1, u_2, ..., u_m) &= 0 \\ &. \\ &. \\ g_n(x_1, x_2, ..., x_n, u_1, u_2, ..., u_m) &= 0 \end{aligned} \tag{2}$$

Note that if m variables u_1, $u_2, ...,$ u_m could be found, the remaining n variables x_1, $x_2, ...,$ x_n are fixed by the n constraints.

In compact form, the above problem can be stated as:

$$minimize\ f(x,u) \tag{3}$$

subject to

$$g(x,u) = 0 \tag{4}$$

where $x \in \mathcal{R}^n$ and $u \in \mathcal{R}^m$

Approach # 1:

- Eliminate the constraint equations by substituting x in terms of u in the function to the optimized.
- Use unconstrained optimization methods to optimize the function in u.

Approach # 2:

- Develop necesary and sufficient conditions of optimality from "scratch" for problems with equality constraints.

Example 1:

$$min\ f(x,u) = 4x^2 + 5u^2 \tag{5}$$

subject to

$$2x + 3u = 6 \tag{6}$$

1. Write x in terms of u in the constraint equation.

$$x = \frac{6 - 3u}{2} \tag{7}$$

2. Substitute x in terms of u in the function to be minimized.

$$\begin{aligned} min\ f(u) &= 4\left(\frac{6-3u}{2}\right)^2 + 5u^2 \\ &= 14u^2 - 36u + 36 \end{aligned} \tag{8}$$

3. Use method developed for minimization of unconstrained functions.

$$\frac{\partial f}{\partial u} = 0 \tag{9}$$

Solving for u, we get:

$$u^* = 1.286 \tag{10}$$

Substitute this value of u in the constraint equation to get the optimal value of x.

$$x^* = 1.071 \tag{11}$$

The minimum value of the function is

$$f(x^*, u^*) = 12.857 \tag{12}$$

Note that if there was no equality constraint, the minimum value of the function would have been ZERO.

Example 2:

$$min\ f(x,u) = (u_1 - x_1)^2 + (x_1 - u_2)^2 + (u_2 - x_2)^4 + (u_2 - x_3)^4 \tag{13}$$

subject to

$$\begin{aligned} x_1 + 3x_2 + 2u_1 - 6 &= 0 \\ 2x_2 + u_1 + 3u_2 - 6 &= 0 \\ x_2 + 3x_3 + 2u_2 - 6 &= 0 \end{aligned} \tag{14}$$

It is not so easy to use Approach # 1 here.

Approach # 1

Advantages:

- Utilizes methods developed for unconstrainted optimization.

Disdvantages:

- It is tedious to eliminate constraints if there are many constraint equations.

Approach # 2: Necessary Conditions

$$minimize\ f(x,u) \tag{15}$$

subject to

$$g(x,u) = 0 \tag{16}$$

where $x \in \mathcal{R}^n$ and $u \in \mathcal{R}^m$

A stationary point is one where $df = 0$ for arbitrary du while holding $dg = 0$ letting dx change as it will.

$$df = \frac{\partial f}{\partial x}.dx + \frac{\partial f}{\partial u}.du \tag{17}$$

and

$$dg = \frac{\partial g}{\partial x}.dx + \frac{\partial g}{\partial u}.du \tag{18}$$

Since $dg = 0$, this implies that

$$dx = -\left[\frac{\partial g}{\partial x}\right]^{-1} \frac{\partial g}{\partial u}.du \tag{19}$$

Substituting in the expression for df, we get

$$df = \left[-\frac{\partial f}{\partial x}\left[\frac{\partial g}{\partial x}\right]^{-1} \frac{\partial g}{\partial u} + \frac{\partial f}{\partial u}\right] du \tag{20}$$

If df has to be zero for *arbitrary* du, it is necessary that

$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial x}\left[\frac{\partial g}{\partial x}\right]^{-1} \frac{\partial g}{\partial u} = 0 \tag{21}$$

Thus, we need to solve *simultaneously*, the following equations:

$$
\begin{aligned}
g(x, u) &= 0 \quad (n \ equations) \\
\frac{\partial f}{\partial u} - \frac{\partial f}{\partial x}\left[\frac{\partial g}{\partial x}\right]^{-1}\frac{\partial g}{\partial u} &= 0 \quad (m \ equations)
\end{aligned}
\tag{22}
$$

Can you solve Example 2 now?

Example 3:

Find the stationary value of

$$f(x,u) = \frac{1}{2}\left(\frac{x^2}{a^2} + \frac{u^2}{b^2}\right) \tag{23}$$

subject to

$$g(x,u) = x + mu - c = 0 \tag{24}$$

Example 4:

A chemical company owns an elliptic piece of land whose principle axes are of lengths $2a$ and $2b$ meters. It is desired to build a *rectangular* tank that has the largest possible perimeter that fits in this land. What are the dimensions of this tank?

Hint: This problem can be mathematically formulated as follows:

$$Maximize\ P = 4(x + y) \tag{25}$$

with the constraint

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{26}$$

Approach # 2

<u>Advantages</u>:

- It is not necessary to eliminate the constraint equations

<u>Disdvantages</u>:

- A large number $(n + m)$ of nonlinear equations have to solved *simultaneously*.

The Case for Using Numerical Methods

- For both unconstrained as well as equality constrained optimization problems, a set of nonlinear algebraic equations have to solved *simultaneously*.
- For all but the simplest problems, these equations are tedious to solve by "hand calculations".

Numerical Problem Formulation

Given a set of n equations in n variables:

$$
\begin{aligned}
p_1(x_1, x_2, ..., x_n) &= 0 \\
p_2(x_1, x_2, ..., x_n) &= 0 \\
&. \\
&. \\
p_n(x_1, x_2, ..., x_n) &= 0
\end{aligned}
\tag{27}
$$

to find a numerical solution for x starting from

$$
x_0 = [x_{10}\ x_{20}\ ...\ x_{n0}]^T \tag{28}
$$

In compact form, we need to find the solution of

$$p(x) = 0 \tag{29}$$

starting from $x = x_0$

Newton's Method

Define $H = \frac{\partial p}{\partial x}|_{x=x_k}$ where x_k is the value of x at the k^{th} iteration.

$$p(x) = p(x_k) + H.(x - x_k) + higher\ order\ terms$$

When $x = x^*$, $p(x^*) = 0$. Substituting above:

$$0 \approx p(x_k) + H(x^* - x_k)$$

Solving for x^*, we get:

$$x^* \approx x_k - (H)^{-1} p(x_k)$$

The above expression is used to develop the Newton's method:

$$x_{k+1} = x_k - (H)^{-1} p(x_k) \tag{30}$$

Can you solve Example 2 now?

Example 5:

Use Newton's method to solve the following system of nonlinear equations:

$$\begin{aligned} 3x_1 - cos(x_2x_3) - \frac{1}{2} &= 0 \\ x_1^2 - 81(x_2 + 0.1)^2 + sin(x_3) + 1.06 &= 0 \\ e^{-x_1x_2} + 20x_3 + \frac{10\pi - 3}{3} &= 0 \end{aligned} \tag{31}$$

starting from the initial condition

$$x_0 = [0.1\ 0.1\ \ -0.1]^T \tag{32}$$

Using the formula

$$x_{k+1} = x_k - (H)^{-1} p(x_k) \tag{33}$$

we get the following:

$$p(x_k) = \begin{bmatrix} 3x_1 - cos(x_2 x_3) - \dfrac{1}{2} \\ x_1^2 - 81(x_2 + 0.1)^2 + sin(x_3) + 1.06 \\ e^{-x_1 x_2} + 20x_3 + \dfrac{10\pi - 3}{3} \end{bmatrix}_{x=x_k}$$

$$H = \begin{bmatrix} 3 & x_3 sin(x_2 x_3) & x_2 sin(x_2 x_3) \\ 2x_1 & -162(x_2 + 0.1) & cos(x_3) \\ -x_2 e^{-x_1 x_2} & -x_1 e^{-x_1 x_2} & 20 \end{bmatrix}_{x=x_k}$$

The results using Newton's method are as follows:

k	x_{1_k}	x_{2_k}	x_{1_k}	$\|\|x_{k+1} - x_k\|\|$
0	0.10000000	0.10000000	-0.10000000	-
1	0.50003702	0.01946686	-0.52152047	0.422
2	0.50004593	0.00158859	-0.52355711	1.79×10^{-2}
3	0.50000034	0.00001244	-0.52359845	1.58×10^{-3}
4	0.50000000	0.00000000	-0.52359877	1.24×10^{-5}