Problems with Inequality Constraints

- In the previous lecture we saw that problems with equality constraints could be handled by modifying methods developed for *unconstrainted* optimization.

- In this lecture, the optimization methods developed in the previous lecture can be further modified to solve the problem with inequality constraints.

Problem Formulation

$$minimize\ f(x_1, x_2, ..., x_n) \tag{1}$$

subject to

$$\begin{aligned} g_1(x_1, x_2, ..., x_n) &= 0 \\ &. \\ g_p(x_1, x_2, ..., x_n) &= 0 \end{aligned} \tag{2}$$

$$\begin{aligned} h_1(x_1, x_2, ..., x_n) &\leq 0 \\ &. \\ h_r(x_1, x_2, ..., x_n) &\leq 0 \end{aligned} \tag{3}$$

Note that there are p equality constraints and r inequality constraints

In compact form, the above problem can be stated as:

$$minimize\ f(x) \tag{4}$$

subject to

$$\begin{aligned} g(x) &= 0 \\ h(x) &\leq 0 \end{aligned} \tag{5}$$

Analysis of Inequality Constraints:

Since $h(x) \leq 0$:

- At the stationary point $x = x^*$, some of the constraints $h_i(x^*) = 0$. These constraints are called *active* constraints.

- At the stationary point $x = x^*$, some of the constraints $h_i(x^*) < 0$. These constraints are called *inactive* constraints.

If we knew *a priori* which constraints were inactive, we could remove them from the problem and then solve the remaining problem with equality constraints using methods developed in the previous lecture.

Algorithm for Handling Inequality Constraints

1. Choose a set of active constraints.
2. Remove the inactive constraints from the problem and solve the resulting problem of optimization with equality constraints.
3. Check and see if the inactive constraints are satisfied. If not, clearly the set of active constraints chosen was incorrect. Go back to step 1.
4. If the inactive constraints are satisfied, compute the value of the objective function and go back to step 1 till all possible combinations of active constraints have been chosen.

5. Find the minimum value of the function among the feasible solutions.

Example 1:

$$Min\ f(x) = (x_1 - 0.5)^2 + (x_2 - 2.5)^2 \tag{6}$$

subject to:

$$\begin{aligned} x_2 - x_1 &\leq 0 \\ -x_1 &\leq 0 \\ -x_2 &\leq 0 \\ x_2 &\leq 2 \end{aligned} \tag{7}$$

Active Const.	x_1	x_2	F	Inactive Const.
None	0.5	2.5	0	Violated
1st	1.5	1.5	2	Not Violated
2nd	0	2.5	0.25	Violated
3rd	0.5	0	6.25	Not Violated
4th	0.5	2.0	0.25	Violated
1st, 2nd	0	0	6.5	Not Violated
1st, 3rd	0	0	6.5	Not Violated
1st, 4th	2	2	2.5	Not Violated
.	.	.	.	
1st, 2nd, 3rd, 4th	-	-	-	Infeasible

A Case for Using Numerical Methods

- For optimization problems with *inequality* constraints, a large number of optimization sub-problems involving *equality* constraints have to be solved.

- Clearly, this is efficiently done using computer methods.

- In MATLAB, there is a routine called ”fmincon” that solves such problems efficiently.

Example 2:

$$min\ F(x) = x_1^2 + x_2^2 + x_3^2 \tag{8}$$

subject to

$$\begin{aligned} 2x_1 &\le 5 \\ x_1 + x_3 &\le 2 \\ x_1 + 2x_2 + x_3 &\le 10 \\ x_1 &\ge 1 \\ x_2 &\ge 2 \\ x_3 &\ge -1 \end{aligned} \tag{9}$$

$$\begin{aligned} 2x_1 - 2x_2 + x_3 &= -2 \\ 10x_1 + 8x_2 - 14x_3 &= 26 \\ -4x_1 + 5x_2 - 6x_3 &= 6 \end{aligned} \tag{10}$$

Example 3:

$$min\ F(x) = x_1^2 + x_2^2 \tag{11}$$

subject to

$$\begin{aligned} x_1 x_2 &\geq 6 \\ x_1 &\geq 2 \\ x_2 &\leq 3 \end{aligned} \tag{12}$$

$$\begin{aligned} 2x_1 - 2x_2 + x_3 &= -2 \\ 10x_1 + 8x_2 - 14x_3 &= 26 \\ -4x_1 + 5x_2 - 6x_3 &= 6 \end{aligned} \tag{13}$$

Example 4:

$$min\ F(x) = (x_1 - 2)^2 + (x_2 - 3)^2 \tag{14}$$

subject to

$$x_1 + x_2 \quad = \quad 3 \tag{15}$$