Exponential of a Matrix

The quantity e^{at} where a and t are *scalar* is an infinite series defined as:

$$e^{at} = 1 + \frac{at}{1!} + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + ... + \frac{a^kt^k}{k!} + ... \qquad (1)$$

When we have a matrix A instead of a scalar a, we can form an infinite series similar to the above equation and use it as a definition of the exponential of a matrix.

$$e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + ... + \frac{A^kt^k}{k!} + ... \qquad (2)$$

Properties of e^{At}

1. $\left. e^{At} \right]_{t=0} = I$

2. $\frac{d}{dt}\left(e^{At}\right) = A.e^{At} = e^{At}.A$

3. $A.\left(\int_0^t e^{Aj} dj\right) = \left(\int_0^t e^{Aj} dj\right).A = e^{At} - I$

4. $e^{A(t+s)} = e^{At}e^{As}$

5. $e^{At}e^{-At} = e^{-At}e^{At} = I$

Solution of Linear Matrix Differential Equation

Using the properties of e^{At}, we can find the solution of a linear matrix differential equation with constant coefficients.

$$\frac{dX}{dt} - AX = BU \tag{3}$$

Multiplying both sides by e^{-At} from the left, we get

$$e^{-At}\frac{dX}{dt} - e^{-At}AX = e^{-At}BU \tag{4}$$

Or

$$\frac{d}{dt}\left(e^{-At}.X\right) = e^{-At}BU \tag{5}$$

Integrating both sides from time 0 to t, we obtain

$$e^{-At}X(t) - IX(0) = \int_0^t e^{-At}BU(t)dt \tag{6}$$

Solving for $X(t)$, we get

$$X(t) = e^{At}.X(0) + e^{At}.\int_0^t e^{-At}BU(t)dt \tag{7}$$

The solution of the differential equation is given by:

$$X(t) = e^{At}.X(0) + e^{At}.\int_0^t e^{-At} BU(t)dt$$

Note that $X(t)$ ***is a vector of dimension*** n
e^{At} ***is a*** $n\mathbf{x}n$ ***matrix; thus*** $e^{At}.X(0)$ ***is a vector of dimension*** n
B ***is a*** $n\mathbf{x}m$ ***matrix and*** U ***is a vector of dimension*** m***; thus*** $e^{At}.\int_0^t e^{-At} BU(t)dt$ ***is a vector of dimension*** n

If we can calculate e^{At}, we can solve for $X(t)$.

Analytical Calculation of e^{At}

If A is an nxn matrix and

$P_A(\lambda) = \lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + a_n$

is its characteristic polynomial, then

$$e^{At} = \Psi_0(t)I + \Psi_1(t)A + ... + \Psi_{n-1}A^{n-1} \tag{8}$$

where $\Psi_0(t)$, $\Psi_1(t)$, ..., $\Psi_{n-1}(t)$ are scalar functions of time.

The functions $\Psi_i(t)$ can be computed from the eigenvalues of the matrix A and for $n = 2, 3$, this computation can be done easily as shown in the next slide.

Analytical Calculation of e^{At} when $n = 2$

Suppose the eigenvalues of the matrix A are λ_1 and λ_2.

1. **When $\lambda_1 \neq \lambda_2$**

$$\begin{aligned} \Psi_0(t) &= -\frac{\lambda_2}{\lambda_1 - \lambda_2} e^{\lambda_1 t} - \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{\lambda_2 t} \\ \Psi_1(t) &= \frac{1}{\lambda_1 - \lambda_2} e^{\lambda_1 t} + \frac{1}{\lambda_2 - \lambda_1} e^{\lambda_2 t} \end{aligned} \tag{9}$$

$$e^{At} = \Psi_0(t) I + \Psi_1(t) A \tag{10}$$

2. When $\lambda_1 = \lambda_2$

$$\begin{aligned} \Psi_0(t) &= e^{\lambda_1 t} - \lambda_1 t e^{\lambda_1 t} \\ \Psi_1(t) &= t e^{\lambda_1 t} \end{aligned} \tag{11}$$

$$e^{At} = \Psi_0(t) I + \Psi_1(t) A \tag{12}$$

$\boxed{\text{Example 1}}$

Calculate e^{At} for

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

Eigenvalues of A:

$$\begin{aligned} det(\lambda I - A) &= det \begin{bmatrix} \lambda - 1 & -2 \\ -4 & \lambda - 3 \end{bmatrix} \\ &= \lambda^2 - 4\lambda - 5 \\ &= (\lambda - 5)(\lambda + 1) \end{aligned}$$

Thus $\lambda_1 = 5,\ \lambda_2 = -1$.

Substituting in the formulae for $\Psi_0(t)$ and $\Psi_1(t)$

$$\Psi_0 = \frac{1}{6}\left(e^{5t} + 5e^{-t}\right)$$

$$\Psi_1 = \frac{1}{6}\left(e^{5t} - e^{-t}\right)$$

Thus,

$$\begin{aligned} e^{At} &= \Psi_0(t)I + \Psi_1(t)A \\ &= \begin{bmatrix} \frac{1}{3}e^{5t} + \frac{2}{3}e^{-t} & \frac{1}{3}e^{5t} - \frac{1}{3}e^{-t} \\ \\ \frac{2}{3}e^{5t} - \frac{2}{3}e^{-t} & \frac{2}{3}e^{5t} + \frac{1}{3}e^{-t} \end{bmatrix} \end{aligned}$$

Example 2

Calculate e^{At} **for**

$$A = \begin{bmatrix} 1 & 2 \\ -8 & -5 \end{bmatrix}$$

Eigenvalues of A**:**

$$\begin{aligned} det(\lambda I - A) &= det \begin{bmatrix} \lambda - 1 & -2 \\ 8 & \lambda + 5 \end{bmatrix} \\ &= \lambda^2 + 4\lambda + 11 \end{aligned}$$

Thus

$\lambda_1 = -2 + i\sqrt{7}$ $\qquad\qquad$ $\lambda_2 = -2 - i\sqrt{7}$

Substituting in the formulae for $\Psi_0(t)$ and $\Psi_1(t)$

$$\Psi_0 = \frac{(2+i\sqrt{7})e^{(-2+i\sqrt{7})t} - (2-i\sqrt{7})e^{(-2-i\sqrt{7})t}}{2i\sqrt{7}}$$

$$\Psi_1 = \frac{e^{(-2+i\sqrt{7})t} - e^{(-2-i\sqrt{7})t}}{2i\sqrt{7}}$$

Using the formula

$$\begin{aligned} \frac{e^{i\theta} + e^{-i\theta}}{2} &= Cos(\theta) \\ \frac{e^{i\theta} - e^{-i\theta}}{2i} &= Sin(\theta) \end{aligned} \tag{13}$$

we get

$$\begin{aligned}
\Psi_0(t) &= e^{-2t}\left[cos(\sqrt{7}t) + \frac{2}{\sqrt{7}}sin(\sqrt{7}t)\right] \\
\Psi_1(t) &= \frac{e^{-2t}}{\sqrt{7}}sin(\sqrt{7}t)
\end{aligned}$$

Thus

$$e^{At} = \Psi_0(t)I + \Psi_1(t)A$$

$$= e^{-2t}\begin{bmatrix} cos(\sqrt{7}t) + \frac{3}{\sqrt{7}}sin(\sqrt{7}t) & \frac{2}{\sqrt{7}}sin(\sqrt{7}t) \\ -\frac{8}{\sqrt{7}}sin(\sqrt{7}t) & cos(\sqrt{7}t) - \frac{3}{\sqrt{7}}sin(\sqrt{7}t) \end{bmatrix}$$

Analytical Calculation of e^{At} when $n = 2$

Suppose the eigenvalues of the matrix A are complex.

$$\begin{aligned} \lambda_1 &= R + i\Omega \\ \lambda_2 &= R - i\Omega \end{aligned} \tag{14}$$

Then,

$$\begin{aligned} \Psi_0(t) &= \left(cos(\Omega t) - \frac{R}{\Omega} Sin(\Omega t) \right) e^{Rt} \\ \Psi_1(t) &= \left(\frac{1}{\Omega} Sin(\Omega t) \right) e^{Rt} \end{aligned} \tag{15}$$

$$e^{At} = \Psi_0(t) I + \Psi_1(t) A \tag{16}$$