Dynamics of a Linear System

A linear dynamical system in deviation form is represented as

$$\begin{aligned} \frac{dX}{dt} &= AX + BU \\ X(0) &= X_0 \end{aligned} \tag{1}$$

where

$$\begin{aligned} X &= x - x_s \\ U &= u - u_s \end{aligned} \tag{2}$$

We saw in the previous lecture that the solution of the above differential equation is given by

$$X(t) = e^{At}.X(0) + e^{At}.\int_0^t e^{-At}BU(t)dt \qquad (3)$$

It is clear from the above equation that $X(t)$ is affected by:

- **the initial conditions, $X(0)$**
- **the input vector, $U(t)$**

We study these effects separately

Unforced Dynamics

Consider the system represented by eq. (1) in the previous slide. Suppose that:

- $U(t) = 0$ **(the inputs are at their steady state values** u_s**)**
- $X(0) \neq 0$ **(the initial conditions are not at their steady state values)**
- **How does** $X(t)$ **change with time?**
- **Do the system states go to their steady state values,** x_s**?**
- **If so, how long does this take?**

Since

$$X(t) = e^{At}.X(0) + e^{At}.\int_0^t e^{-At}BU(t)dt \tag{4}$$

and it is given that $U(t) = 0$ **but** $X(0) \neq 0$**, we can simplify the above equation to get**

$$X(t) = e^{At}.X(0) \tag{5}$$

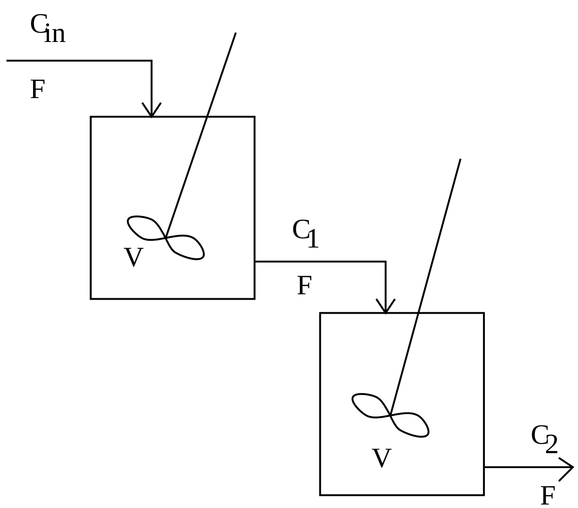
Writing the above equation in the original (non-deviation) variables, $x(t)$**, we get**

$$x(t) = x_s + e^{At}.(x(0) - x_s) \tag{6}$$

The exponential matrix e^{At} **relates the unforced dynamic response to the initial conditions.**

Example 1

There are two mixing tanks (of equal volume V) in series as shown in the figure below.



Example 1

- **We develop a dynamic model for this process and**
- **We apply the linearization formula to this model to get**

$$\frac{d}{dt}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} -\frac{F}{V} & 0 \\ \frac{F}{V} & -\frac{F}{V} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} \frac{F}{V} \\ 0 \end{bmatrix} U \quad (7)$$

where

$$\begin{aligned} X_1 &= C_1 - C_{1s} \\ X_2 &= C_2 - C_{2s} \\ U &= C_{in} - C_{ins} \\ V &= \textbf{Volume of each tank} \end{aligned} \qquad (8)$$

At steady state, there is pure water in both tanks as well as the feed. Thus

$$C_{ins} = C_{1s} = C_{2s} = 0 \qquad (9)$$

Case 1: The first tank is at steady state but the second tank is not.

- $C_{in}(t) = 0$ **(feed is pure water)**
- $C_1(0) = 0$ **(pure water in tank 1)**
- $C_2(0) = 10$ **(there is a colored dye in tank 2)**

How do the concentrations, C_1 and C_2, change with time?

The above situation corresponds to

- $U(t) = 0$
- $X_1(0) = 0$
- $X_2(0) = 10$

The unforced dynamic response is given by:

$$X(t) = e^{At}.X(0) \tag{10}$$

For the system considered in this example,

$$e^{At} = \begin{bmatrix} e^{-\frac{F}{V}t} & 0 \\ \frac{F}{V}te^{-\frac{F}{V}t} & e^{-\frac{F}{V}t} \end{bmatrix} \tag{11}$$

Thus,

$$\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \begin{bmatrix} e^{-\frac{F}{V}t} & 0 \\ \frac{F}{V}te^{-\frac{F}{V}t} & e^{-\frac{F}{V}t} \end{bmatrix} \begin{bmatrix} 0 \\ 10 \end{bmatrix} \tag{12}$$

Thus

$$\begin{aligned} X_1(t) &= 0 \\ X_2(t) &= 10.e^{-\frac{F}{V}t} \end{aligned} \tag{13}$$

Substituting the original variables, $C_1(t)$ **and** $C_2(t)$**, we get**

$$\begin{aligned} C_1(t) - C_{1s} &= 0 \\ C_2(t) - C_{2s} &= 10.e^{-\frac{F}{V}t} \end{aligned} \tag{14}$$

Since $C_{1s} = C_{2s} = 0$**, we get**

$$\begin{aligned} C_1(t) &= 0 \\ C_2(t) &= 10.e^{-\frac{F}{V}t} \end{aligned} \tag{15}$$

Case 2: The second tank is at steady state but the first tank is not.

- $C_{in}(t) = 0$ **(feed is pure water)**
- $C_1(0) = 10$ **(there is a colored dye in tank 1)**
- $C_2(0) = 0$ **(pure water in tank 2)**

How do the concentrations, C_1 and C_2, change with time?

The above situation corresponds to

- $U(t) = 0$
- $X_1(0) = 10$
- $X_2(0) = 0$

The unforced dynamic response is given by:

$$X(t) = e^{At}.X(0) \tag{16}$$

For the system considered in this example,

$$e^{At} = \begin{bmatrix} e^{-\frac{F}{V}t} & 0 \\ \frac{F}{V}te^{-\frac{F}{V}t} & e^{-\frac{F}{V}t} \end{bmatrix} \tag{17}$$

Thus,

$$\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \begin{bmatrix} e^{-\frac{F}{V}t} & 0 \\ \frac{F}{V}te^{-\frac{F}{V}t} & e^{-\frac{F}{V}t} \end{bmatrix} \begin{bmatrix} 10 \\ 0 \end{bmatrix} \tag{18}$$

Thus

$$\begin{aligned} X_1(t) &= 10.e^{-\frac{F}{V}t} \\ X_2(t) &= 10.\frac{F}{V}t.e^{-\frac{F}{V}t} \end{aligned} \tag{19}$$

Substituting the original variables, $C_1(t)$ **and** $C_2(t)$**, we get**

$$\begin{aligned} C_1(t) - C_{1s} &= 10.e^{-\frac{F}{V}t} \\ C_2(t) - C_{2s} &= 10.\frac{F}{V}t.e^{-\frac{F}{V}t} \end{aligned} \tag{20}$$

Since $C_{1s} = C_{2s} = 0$**, we get**

$$\begin{aligned} C_1(t) &= 10.e^{-\frac{F}{V}t} \\ C_2(t) &= 10.\frac{F}{V}t.e^{-\frac{F}{V}t} \end{aligned} \tag{21}$$

Qualitative Characteristics

Will the process states go back to steady state?

In other words, will $\lim_{t \to \infty} X(t) = 0$?

If the answer is yes, we say that the process is stable

If the answer is no, we say that the process is unstable

The above question can be answered by checking the following

$$\lim_{t \to \infty} e^{At} = 0 \tag{22}$$

Remember that in the analytical calculation of e^{At}, we get expressions of the form of $e^{\lambda_i t}$ where λ_i are the eigenvalues of the matrix A. Thus, in order for $\lim_{t \to \infty} e^{At} = 0$, all eigenvalues of the A matrix must have negative real parts.

Stability $\leftrightarrow$ All eigenvalues, λ_i, of the matrix A have negative real parts

Qualitative Characteristics

Will the response be oscillatory or smooth?

An oscillatory response results when there are "sines and cosines" in the solution. These result when some of the eigenvalues of the matrix A are complex.

Oscillatory response $\leftrightarrow$ ***The matrix*** A ***has* atleast *one pair of complex eigenvalues.***
Smooth response $\leftrightarrow$ **All** ***eigenvalues of the matirx*** A ***are real.***