Dynamics of a Linear System

A linear dynamical system in deviation form is represented as

$$\begin{aligned} \frac{dX}{dt} &= AX + BU \\ X(0) &= X_0 \end{aligned} \tag{1}$$

where

$$\begin{aligned} X &= x - x_s \\ U &= u - u_s \end{aligned} \tag{2}$$

We saw in the previously that the solution of the above differential equation is given by

$$X(t) = e^{At}.X(0) + e^{At}.\int_0^t e^{-At} BU(t) dt' \qquad (3)$$

It is clear from the above equation that $X(t)$ is affected by:

- **the initial conditions, $X(0)$**
- **the input vector, $U(t)$**

In the previous lecture, we studied the effect of initial conditions (unforced response). We now study the effect of the input (forced response).

Forced Dynamics

Consider the system represented by eq. (1) in the previous slide. Suppose that:

- $X(0) = 0$ **(the initial conditions are at their steady state values)**
- $U(t) \neq 0$ **(the inputs are not at their steady state values** u_s**)**
- **How does** $X(t)$ **change with time?**

Clearly, the system states will not go to their old steady state values

Since

$$X(t) = e^{At}.X(0) + e^{At}.\int_0^t e^{-At}BU(t)dt \qquad (4)$$

and it is given that $X(0) = 0$ **but** $U(t) \neq 0$**, we can simplify the above equation to get**

$$X(t) = e^{At}.\int_0^t e^{-At}BU(t)dt \qquad (5)$$

Writing the above equation in the original (non-deviation) variables, $x(t)$**, we get**

$$x(t) = x_s + e^{At}.\int_0^t e^{-At}BU(t)dt \qquad (6)$$

Typical Inputs in the Process Industry

1. **Step Input:**

$$u(t) = \begin{cases} u_s & t < 0 \\ u_s + M & t \geq 0 \end{cases} \tag{7}$$

In deviation form:

$$U(t) = \begin{cases} 0 & t < 0 \\ M & t \geq 0 \end{cases} \tag{8}$$

2. Pulse Input:

$$u(t) = \begin{cases} u_s & t < 0 \\ u_s + \dfrac{M}{t_w} & 0 \leq t < t_w \\ u_s & t \geq t_w \end{cases} \tag{9}$$

In deviation form:

$$U(t) = \begin{cases} 0 & t < 0 \\ \dfrac{M}{t_w} & 0 \leq t < t_w \\ 0 & t \geq t_w \end{cases} \tag{10}$$

3. Ramp Input:

$$u(t) = \begin{cases} u_s & t < 0 \\ u_s + M.t & t \geq 0 \end{cases} \tag{11}$$

In deviation form:

$$U(t) = \begin{cases} 0 & t < 0 \\ M.t & t \geq 0 \end{cases} \tag{12}$$

4. Sinusoidal Input:

$$u(t) = \begin{cases} u_s & t < 0 \\ u_s + Msin(\omega t) & t \geq 0 \end{cases} \tag{13}$$

In deviation form:

$$U(t) = \begin{cases} 0 & t < 0 \\ Msin(\omega t) & t \geq 0 \end{cases} \tag{14}$$

Procedure to Calculate Forced Response

Consider a system represented by

$$\frac{dx}{dt} = f(x, u) \tag{15}$$

Given $u(t)$ **for** $t < 0$ **and** $t \geq 0$**, the forced response can be calculated using the following procedure:**

1. **Calculate the steady states** x_s **given** u_s **by solving**

$$f(x_x, u_s) = 0 \tag{16}$$

2. **Linearize the system and put in deviation**

form to get

$$\frac{dX}{dt} = AX + BU \tag{17}$$

3. **Compute** e^{At}
4. **Compute** $e^{-At}BU(t)$
5. **Compute** $\int_0^t e^{-At}BU(t)dt$
6. **Compute** $X(t) = e^{At}\int_0^t e^{-At}BU(t)dt$
7. **Compute** $x(t) = x_s + X(t)$

Example 1

An autocatalytic reaction $A \rightarrow R$ takes place in a constant volume isothermal reactor. Component mass balances give the following reactor model:

$$\begin{aligned} \frac{dC_A}{dt} &= \frac{F}{V}(C_{Ain} - C_A) - kC_AC_R \\ \frac{dC_R}{dt} &= \frac{F}{V}(C_{Rin} - C_R) + kC_AC_R \end{aligned} \tag{18}$$

In the above model

- C_{Ain} **and** C_{Rin} **are the inputs**
- C_A **and** C_R **are the states**
- **At steady state** $C_{Ain_s} = 1$ **and** $C_{Rin_s} = 1$

- $\frac{F}{V} = 1$
- $k = 2$

1. **Find the steady state values of C_A and C_R**
2. **Linearize the model around the steady state and put it in the form**

$$\frac{dX}{dt} = AX + BU \tag{19}$$

3. **Suppose the system states are initially at their steady state values but both inputs suddenly increase to 1.5 from 1.0. How do C_A and C_R change with time ?**

$\boxed{\text{Process Outputs}}$

Very often, we are not interested in calculating the response for the entire state vector, $x(t)$; we may care about only a few variables that are critical to the operation of the process. Such variables are called Outputs.

In general, the outputs are algebraic functions of the states, x and the inputs, u, and are represented by the vector, y.

$$y = g(x, u) \tag{20}$$

At steady state

$$y_s = g(x_s, u_s) \tag{21}$$

We define deviation variables for the outputs as

$$Y = y - y_s \tag{22}$$

Using the linearization formula, a linearized representation of the outputs is given by

$$Y = CX + DU \tag{23}$$

where

$$\begin{aligned} C &= \left[\frac{\partial g}{\partial x}\right]_{x_s, u_s} \\ D &= \left[\frac{\partial g}{\partial u}\right]_{x_s, u_s} \end{aligned} \tag{24}$$

Example 2

Linearize the following nonlinear system around the steady state $(0, 0)$

$$
\begin{aligned}
\frac{dx_1}{dt} &= x_1^2 + x_1 + x_2 + u \\
\frac{dx_2}{dt} &= x_1^2 x_2 + x_2 + u \\
y &= x_2
\end{aligned}
\tag{25}
$$

Solution:

$$
A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}
$$

$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \end{bmatrix}$$

Thus, the linearized system is:

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} U \\ Y &= \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + 0.U \end{aligned} \tag{26}$$

Dynamics of a Linear System

A linear dynamical system in deviation form is represented as

$$\begin{aligned}
\frac{dX}{dt} &= AX + BU \\
Y &= CX + DU \\
X(0) &= X_0
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
X &= x - x_s \\
U &= u - u_s \\
Y &= y - y_s
\end{aligned} \tag{28}$$

We saw in the previously that the dynamics of the states, $X(t)$, is given by

$$X(t) = e^{At}.X(0) + e^{At}.\int_0^t e^{-At} BU(t)dt \qquad (29)$$

The dynamics of the outputs is given by

$$Y(t) = C.e^{At}.X(0) + C.e^{At}.\int_0^t e^{-At} BU(t)dt + D.U \qquad (30)$$

- **Unforced Dynamics:**

$$Y(t) = C.e^{At}.X(0) \tag{31}$$

- **Forced Dynamics:**

$$Y(t) = C.e^{At}.\int_0^t e^{-At} BU(t)dt + D.U \tag{32}$$