Illustrative Example

Consider the following system in deviation form:

$$\frac{d}{dt}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} U$$

Assume that $X(0) = 0$.

Calculate the forced response to the following input:

$$U = \begin{cases} 0 & t < 0 \\ 0.5 & 0 \leq t < 1 \\ 0 & t \geq 1 \end{cases}$$

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$

Thus,

$$-A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Using the analytical formula for calculating e^{At}, we get:

$$e^{-At} = \begin{bmatrix} e^{t} & 0 \\ -te^{t} & e^{t} \end{bmatrix}$$

Since

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

we have

$$\begin{aligned} e^{-At}B &= \begin{bmatrix} e^{t} & 0 \\ -te^{t} & e^{t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} e^{t} \\ -te^{t} \end{bmatrix} \end{aligned}$$

Now,

$$e^{At} = \begin{bmatrix} e^{-t} & 0 \\ te^{-t} & e^{-t} \end{bmatrix}$$

Thus:

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ te^{-t} & e^{-t} \end{bmatrix} \int_0^t \begin{bmatrix} e^{t} \\ -te^{t} \end{bmatrix} U(t)dt$$

Solution for $0 \leq t < 1$

$$X(t) \;=\; e^{At}\int_0^t e^{-At}B\,\underbrace{U(t)}_{U=0.5}\,dt$$

$$
\begin{aligned}
\int_0^t e^{-At} BU\,dt &= \begin{bmatrix} \int_0^t 0.5e^t\,dt \\ \int_0^t -0.5te^t\,dt \end{bmatrix} \\
&= \begin{bmatrix} 0.5e^t - 0.5 \\ -0.5te^t + 0.5e^t - 0.5 \end{bmatrix}
\end{aligned}
$$

Remember to apply the limits of integration.

Thus

$$
\begin{aligned}
\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} &= e^{At}\int_0^t e^{-At}BU(t)dt \\
&= \begin{bmatrix} e^{-t} & 0 \\ te^{-t} & e^{-t} \end{bmatrix} \begin{bmatrix} 0.5e^t - 0.5 \\ -0.5te^t + 0.5e^t - 0.5 \end{bmatrix} \\
&= \begin{bmatrix} 0.5(1 - e^{-t}) \\ 0.5(1 - e^{-t} - te^{-t}) \end{bmatrix}
\end{aligned}
$$

Thus when $0 \leq t < 1$,

$$
\begin{aligned}
X_1 &= 0.5(1 - e^{-t}) \\
X_2 &= 0.5(1 - e^{-t} - te^{-t})
\end{aligned}
\tag{1}
$$

Solution for $t \geq 1$

$$X(t) \; = \; e^{At} \int_0^t e^{-At} BU(t) dt$$

The integration is from 0 to t where $t \geq 1$. We split the integral into two parts as shown below:

$$\begin{aligned} \int_0^t e^{-At} BU(t) dt &= \int_0^1 e^{-At} B \underbrace{U(t)}_{U=0.5} dt + \int_1^t e^{-At} B \underbrace{U(t)}_{U=0} dt \\ &= \int_0^1 e^{-At} B \underbrace{U(t)}_{U=0.5} dt \end{aligned}$$

$$
\begin{aligned}
\int_0^1 e^{-At} B U(t) dt &= \begin{bmatrix} 0.5e^t]_0^1 \\ -0.5(te^t - e^t)]_0^1 \end{bmatrix} \\
&= \begin{bmatrix} 0.5e - 0.5 \\ -0.5 \end{bmatrix} \\
&= \begin{bmatrix} 0.859 \\ -0.5 \end{bmatrix}
\end{aligned}
$$

Thus,

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = e^{At}\int_0^t e^{-At}BU(t)dt$$
$$= \begin{bmatrix} e^{-t} & 0 \\ te^{-t} & e^{-t} \end{bmatrix} \begin{bmatrix} 0.859 \\ -0.5 \end{bmatrix}$$
$$= \begin{bmatrix} 0.859e^{-t} \\ 0.859te^{-t} - 0.5e^{-t} \end{bmatrix}$$

Thus when $t \geq 1$,

$$\begin{aligned} X_1 &= 0.859e^{-t} \\ X_2 &= 0.859te^{-t} - 0.5e^{-t} \end{aligned} \tag{2}$$

What are X_1 and X_2 when $t = 0.4$?

Substitute $t = 0.4$ in equation 1 to get:

$$\begin{aligned} X_1 &= 0.165 \\ X_2 &= 0.031 \end{aligned}$$

What are X_1 and X_2 when $t = 1.3$?

Substitute $t = 1.3$ in equation 2 to get:

$$\begin{aligned} X_1 &= 0.234 \\ X_2 &= 0.168 \end{aligned}$$