Interconnected Systems

So far in this course, we have done the following:

- **Developed dynamic models for chemical processes**
- **Studied the effect of changing inputs on the outputs and states**

In this lecture, we will address the issue of what happens when several systems are interconnected.

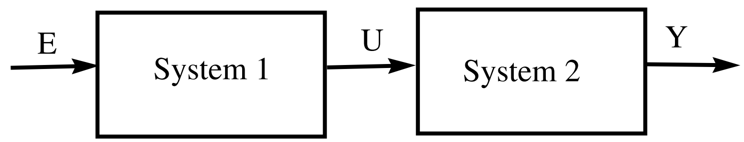
In particular, we will show that if each of the sub-systems is of the form

$$\begin{aligned} \dot{X} &= AX + BU \\ Y &= CX + DU \end{aligned} \tag{1}$$

then the overall interconnected system is of the same form as eq. (1).

Series Connection

Consider the following system in *series*:



where

System 1:

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} &= \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} E \\ U &= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \end{aligned} \tag{2}$$

System 2:

$$
\begin{aligned}
\frac{d}{dt}\begin{bmatrix} X_3 \\ X_4 \end{bmatrix} &= \begin{bmatrix} -2 & 0 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} U \\
Y &= \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} X_3 \\ X_4 \end{bmatrix}
\end{aligned} \tag{3}
$$

What is the overall system with E as the input and Y as the output.

<u>Solution</u>:

- **Step 1: Put the dynamics in ”non-matrix” form.**

$$
\begin{aligned}
\frac{dX_1}{dt} &= -X_1 + 2X_2 + E \\
\frac{dX_2}{dt} &= -3X_2 \\
\frac{dX_3}{dt} &= -2X_3 + U \\
\frac{dX_4}{dt} &= -4X_3 - 5X_4 + U \\
U &= X_1 \\
Y &= X_3 + X_4
\end{aligned} \tag{4}
$$

- **Step 2: Eliminate the variable U by substitution.**

$$\begin{aligned}
\frac{dX_1}{dt} &= -X_1 + 2X_2 + E \\
\frac{dX_2}{dt} &= -3X_2 \\
\frac{dX_3}{dt} &= -2X_3 + X_1 \\
\frac{dX_4}{dt} &= -4X_3 - 5X_4 + X_1 \\
Y &= X_3 + X_4
\end{aligned} \tag{5}$$

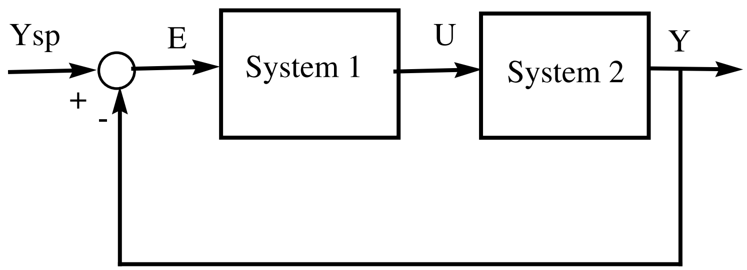
- **Step 3: Put the system back in matrix form.**

$$\frac{d}{dt}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 4 & -5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} E$$

$$Y = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + [0]E \tag{6}$$

Error Feedback Connection

Consider the following system in *series*:



where System 1 and System 2 were described previously. Furthermore,

$$E = Y_{sp} - Y \qquad (7)$$

What is the overall system with Y_{sp} as the input and Y as the output.

<u>Solution</u>:

- **Step 1: Put dynamics in "non-matrix" form.**

$$\begin{aligned}
\frac{dX_1}{dt} &= -X_1 + 2X_2 + E \\
\frac{dX_2}{dt} &= -3X_2 \\
\frac{dX_3}{dt} &= -2X_3 + U \\
\frac{dX_4}{dt} &= -4X_3 - 5X_4 + U \qquad (8) \\
U &= X_1 \\
Y &= X_3 + X_4 \\
E &= Y_{sp} - Y \\
&= Y_{sp} - X_3 - X_4
\end{aligned}$$

- **Step 2: Eliminate the variables U and E by substitution.**

$$
\begin{aligned}
\frac{dX_1}{dt} &= -X_1 + 2X_2 + Y_{sp} - X_3 - X_4 \\
\frac{dX_2}{dt} &= -3X_2 \\
\frac{dX_3}{dt} &= -2X_3 + X_1 \\
\frac{dX_4}{dt} &= -4X_3 - 5X_4 + X_1 \\
Y &= X_3 + X_4
\end{aligned}
\tag{9}
$$

- **Step 3: Put the system back in matrix form.**

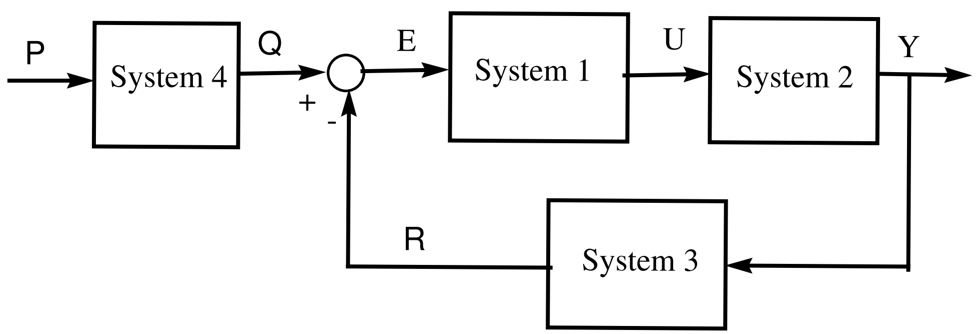
$$\frac{d}{dt}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -1 & -1 \\ 0 & -3 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 4 & -5 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} Y_{sp}$$

$$Y = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + [0]Y_{sp} \quad (10)$$

More Complicated Structures

More complicated structures can be analyzed using the same procedure.

Consider the following interconnected system



where System 1 and System 2 are the same as the ones described previously and Systems 3 and 4 are as follows:

System 3:

$$\frac{d}{dt}\begin{bmatrix} X_5 \\ X_6 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} X_5 \\ X_6 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} Y$$
$$R = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} X_5 \\ X_6 \end{bmatrix} + 5E \tag{11}$$

System 4:

$$\frac{d}{dt}\begin{bmatrix} X_7 \\ X_8 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} X_7 \\ X_8 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} P$$
$$Q = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} X_7 \\ X_8 \end{bmatrix} \tag{12}$$

Furthermore:

$$E = Q - R \tag{13}$$

What is the overall system with P as the input and Y as the output?