Interconnected Systems

So far in this course, we have done the following:

- Developed dynamic models for chemical processes
- Studied the effect of changing inputs on the outputs and states

In this lecture, we will address the issue of what happens when several systems are interconnected.

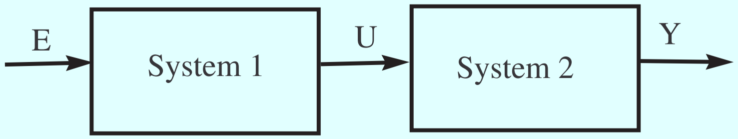
In particular, we will show that if each of the sub-systems is of the form

$$\begin{aligned} \dot{X} &= AX + BU \\ Y &= CX + DU \end{aligned} \tag{1}$$

then the overall interconnected system is of the same form as eq. (1).

Series Connection

Consider the following system in *series*:



where

System 1:

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} &= \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} E \\ U &= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \end{aligned} \tag{2}$$

System 2:

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix} X_3 \\ X_4 \end{bmatrix} &= \begin{bmatrix} -2 & 0 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} U \\ Y &= \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} X_3 \\ X_4 \end{bmatrix}\end{aligned} \tag{3}$$

What is the overall system with E as the input and Y as the output.

Solution:

- Step 1: Put the dynamics in "non-matrix" form.

$$\begin{aligned}
\frac{dX_1}{dt} &= -X_1 + 2X_2 + E \\
\frac{dX_2}{dt} &= -3X_2 \\
\frac{dX_3}{dt} &= -2X_3 + U \\
\frac{dX_4}{dt} &= -4X_3 - 5X_4 + U \\
U &= X_1 \\
Y &= X_3 + X_4
\end{aligned} \tag{4}$$

- Step 2: Eliminate the variable U by substitution.

$$\begin{aligned}
\frac{dX_1}{dt} &= -X_1 + 2X_2 + E \\
\frac{dX_2}{dt} &= -3X_2 \\
\frac{dX_3}{dt} &= -2X_3 + X_1 \\
\frac{dX_4}{dt} &= -4X_3 - 5X_4 + X_1 \\
Y &= X_3 + X_4
\end{aligned} \tag{5}$$

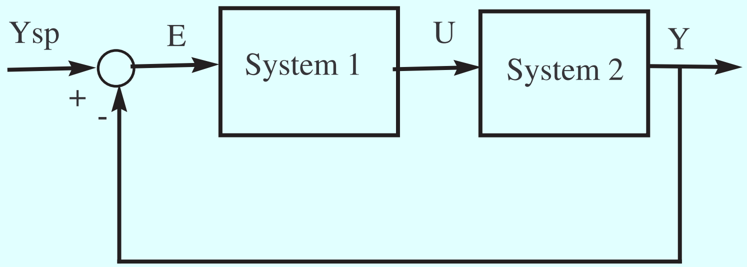
- Step 3: Put the system back in matrix form.

$$\frac{d}{dt}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 4 & -5 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + [0]E \tag{6}$$

Error Feedback Connection

Consider the following system in *series*:



where System 1 and System 2 were described previously. Furthermore,

$$E = Y_{sp} - Y \tag{7}$$

What is the overall system with Y_{sp} as the input and Y as the output.

Solution:

- Step 1: Put the dynamics in "non-matrix" form.

$$\begin{aligned}
\frac{dX_1}{dt} &= -X_1 + 2X_2 + E \\
\frac{dX_2}{dt} &= -3X_2 \\
\frac{dX_3}{dt} &= -2X_3 + U \\
\frac{dX_4}{dt} &= -4X_3 - 5X_4 + U \\
U &= X_1 \\
Y &= X_3 + X_4 \\
E &= Y_{sp} - Y \\
&= Y_{sp} - X_3 - X_4
\end{aligned} \tag{8}$$

- Step 2: Eliminate the variables U and E by substitution.

$$
\begin{aligned}
\frac{dX_1}{dt} &= -X_1 + 2X_2 + Y_{sp} - X_3 - X_4 \\
\frac{dX_2}{dt} &= -3X_2 \\
\frac{dX_3}{dt} &= -2X_3 + X_1 \\
\frac{dX_4}{dt} &= -4X_3 - 5X_4 + X_1 \\
Y &= X_3 + X_4
\end{aligned} \tag{9}
$$

- Step 3: Put the system back in matrix form.

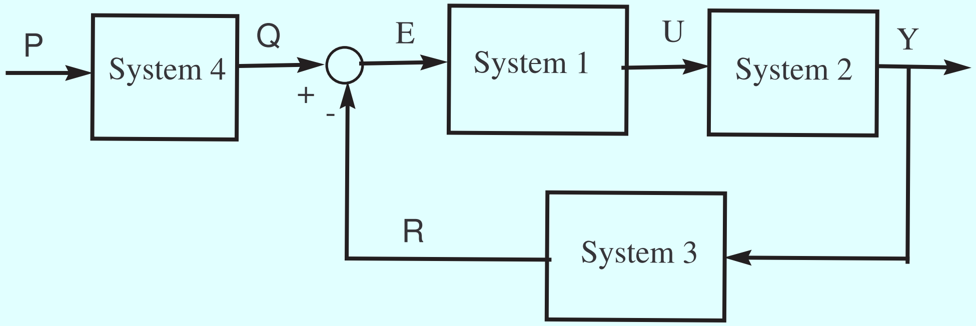
$$\frac{d}{dt}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -1 & -1 \\ 0 & -3 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 4 & -5 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + [0]Y_{sp} \tag{10}$$

More Complicated Structures

More complicated structures can be analyzed using the same procedure.

Consider the following interconnected system



where System 1 and System 2 are the same as the ones described previously and Systems 3 and 4 are as follows:

System 3:

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} X_5 \\ X_6 \end{bmatrix} &= \begin{bmatrix} 8 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} X_5 \\ X_6 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} Y \\ R &= \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} X_5 \\ X_6 \end{bmatrix} + 5E \end{aligned} \tag{11}$$

System 4:

$$\begin{aligned}
\frac{d}{dt}\begin{bmatrix} X_7 \\ X_8 \end{bmatrix} &= \begin{bmatrix} -3 & 5 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} X_7 \\ X_8 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} P \\
Q &= \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} X_7 \\ X_8 \end{bmatrix}
\end{aligned} \tag{12}$$

Furthermore:

$$E = Q - R \tag{13}$$

What is the overall system with P as the input and Y as the output?