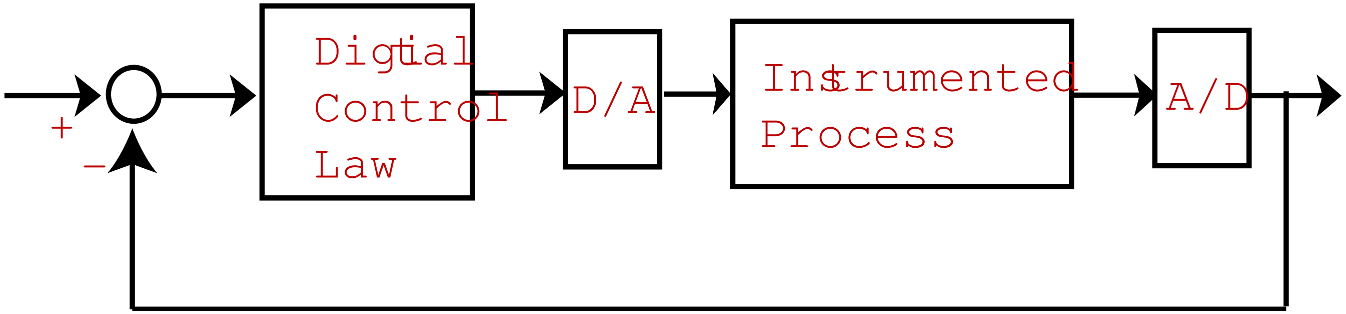
Digital Control Loop

- **The material covered so far has been based on continuous-time model for the process and a continuous-time control law.**
- **However, modern control technology is computer-based which operates in discrete time.**

In this lecture, we develop a systematic approach for implementing controllers in a digital control loop.

The digital control loop is shown below:



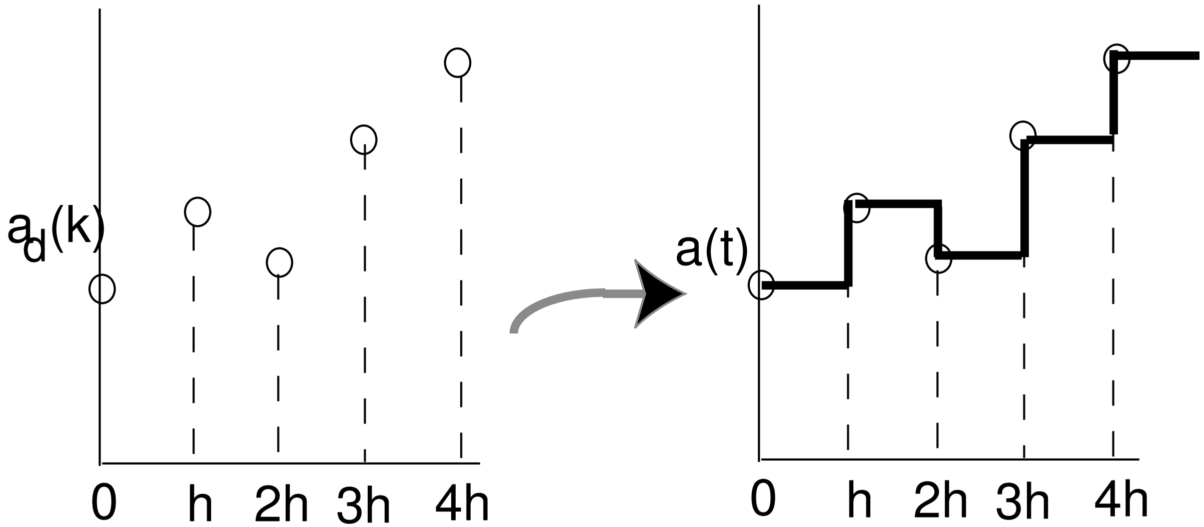
There are two additional hardware elements:

1. **Digital to Analog Converter (D/A)**

2. **Analog to Digital Converter (A/D)**

D/A Converter

- **The D/A converter reconstructs an analog signal from a digital**
- **This is done via a zero-order hold: the analog signal is held constant between sampling instants as shown below.**

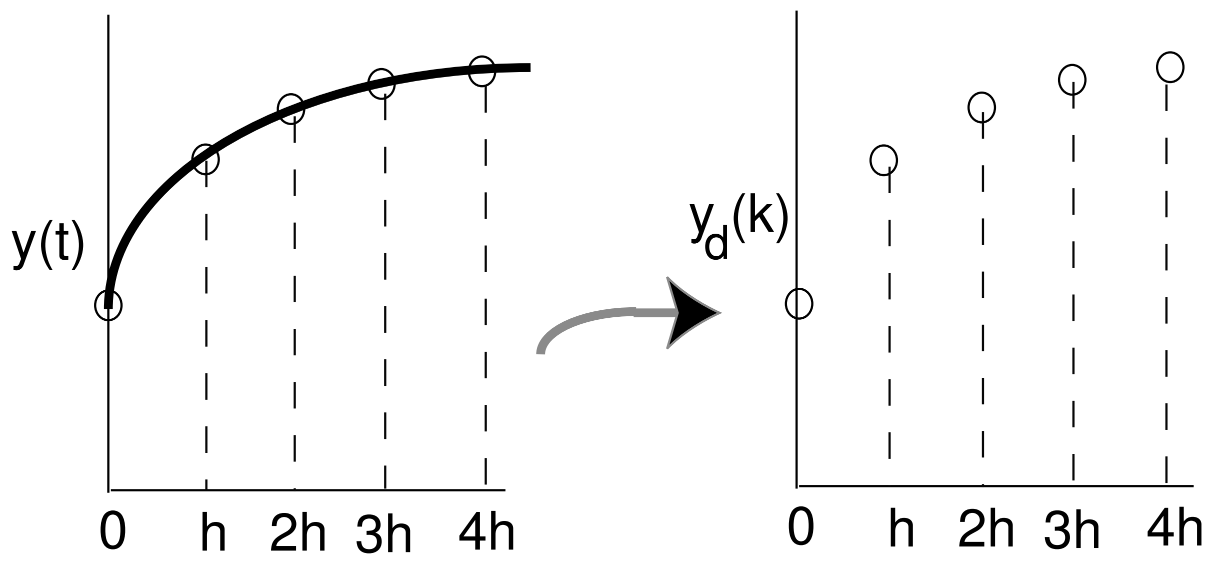


- **Mathematically, the zero-order hold operation can be expressed as:**

$$a(t) = a_d(k-1) \qquad for\ (k-1)h \leq t \leq kh \quad (1)$$

A/D Converter

- **The A/D converter is a sampler: given an analog signal, it samples its values at different time instants and generates a discrete signal as shown in the figure below:**



- **Mathematically, the sampler operation can be expressed as:**

$$y_d(k) = y(kh) \qquad for\ k = 0, 1, 2, 3, ... \tag{2}$$

Two Points of View

- **VIEW 1: Analysis/Design is first performed in continuous time. Then the designed continuous time controller (e.g. PID) is approximately discretized so that the computer can implement this control law.**

- **VIEW 2: The process is exactly discretized first. Then, a discrete controller is designed that can be implemented by the computer.**

No matter which point of view is followed, one has to develop a method for discretizing linear systems of the form:

$$\begin{aligned} \frac{dX}{dt} &= AX + BU \\ Y &= CX + DU \end{aligned} \tag{3}$$

Discretization Formula

Consider a system of the form

$$\begin{aligned} \frac{dX}{dt} &= AX + BU \\ Y &= CX + DU \end{aligned} \tag{4}$$

If the input $U(t)$ is discretized as $U(t) = U(k-1)$, then the dynamic system is discretized as:

$$\begin{aligned} X(k) &= e^{Ah} X(k-1) + \left(\int_0^h e^{At} B dt \right) U(k-1) \\ Y(k) &= CX(k) + DU(k) \end{aligned} \tag{5}$$

Example

Consider the following dynamic system

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} U(t) \\ Y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\end{aligned} \tag{6}$$

If the input $U(t)$ is discretized as $U(t) = U(k-1)$, then what is discretized version of the dynamic system?