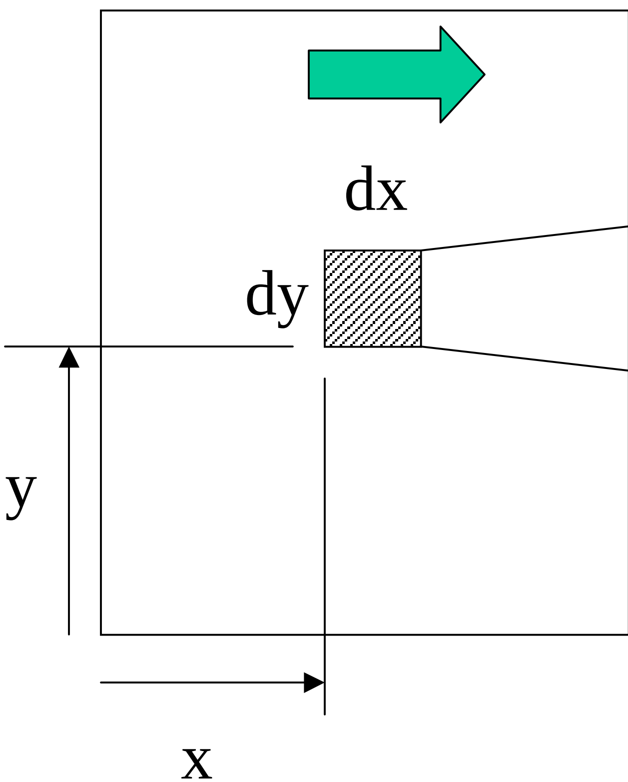
Heat Diffusion Equation

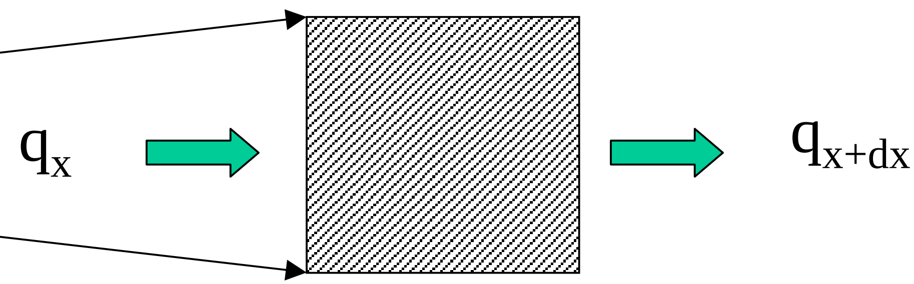
All go to zero

Energy balance equation: $\frac{dE}{dt} = \frac{dE_g}{dt} + q_{in} - q_{out} + \dot{E}_1 - \dot{E}_2 - \frac{dW}{dt}$

Apply this equation to a solid undergoing conduction heat transfer:



$E = mc_pT = (\rho V)c_pT = \rho(dxdydz)c_pT$



$$q_x = -KA\left.\frac{\partial T}{\partial x}\right|_x = -K(dydz)\left.\frac{\partial T}{\partial x}\right|_x$$

$$q_{x+dx} = q_x + dq_x = q_x + \frac{\partial q_x}{\partial x}dx$$

Heat Diffusion Equation (2)

Energy Storage = Energy Generation + Net Heat Transfer

$$\frac{\partial}{\partial t}\left(\rho c_p T\right) dxdydz = \dot{q}\, dxdydz + q_x - q_{x+dx}$$

$$\rho c_p \frac{\partial T}{\partial t} dxdydz = \dot{q}\, dxdydz + q_x - (q_x + \frac{\partial q_x}{\partial x} dx)$$

$$= \dot{q}\, dxdydz - \frac{\partial}{\partial x}(-k\frac{\partial T}{\partial x}) dxdydz$$

$$\rho c_p \frac{\partial T}{\partial t} = \dot{q} + \frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial T}{\partial y}) + \frac{\partial}{\partial z}(k\frac{\partial T}{\partial z})$$

Note: partial differential operator is used since T=T(x,y,z,t)

Generalized to three-dimensional

Heat Diffusion Equation (3)

$$\rho c_p \frac{\partial T}{\partial t} = \dot{q} + \frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial T}{\partial y}) + \frac{\partial}{\partial z}(k\frac{\partial T}{\partial z})$$

Special case1: no generation $\dot{q} = 0$

Special case 2: constant thermal conductivity k = constant

$$\rho c_p \frac{\partial T}{\partial t} = k(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}) + \dot{q} = k\nabla^2 T + \dot{q},$$

$$where\, \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ is the Laplacian operator

Special case 3: $\frac{\partial}{\partial \mathrm{t}} = 0$ and $\dot{q} = 0$

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0,$$ The famous Laplace' s equation

1-D, Steady Heat Transfer

Assume steady and no generation, 1 - D Laplace's equation

$$\frac{d^2T}{dx^2} = 0, \; T(x,y,x,t) = T(x), \text{ function of x coordinate alone}$$

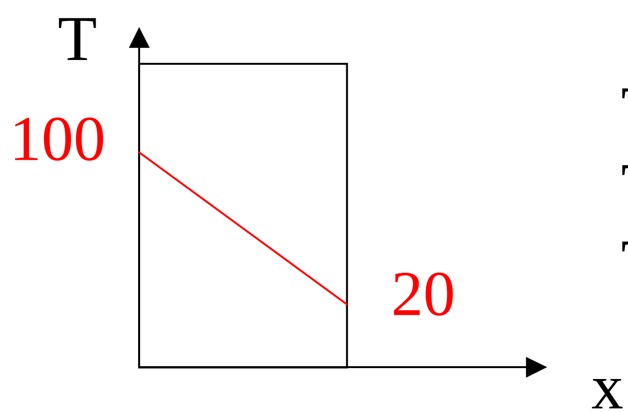
Note: ordinary differential operator is used since T = T(x) only

The general solution of this equation can be determined by integting twice:

First integration leads to $\frac{dT}{dx} = constant = C_1$.

Integrate again $T(x) = C_1 x + C_2$

Second order differential equation: need two boundary conditions to determine the two constants C_1 and C_2.



$T(x=0)=100°C=C_2$

$T(x=1\text{ m})=20°C=C_1+C_2$, $C_1=-80°C$

$T(x)=100-80x$ (°C)

1-D Heat Transfer (cont.)

Recall Fourier's Law:

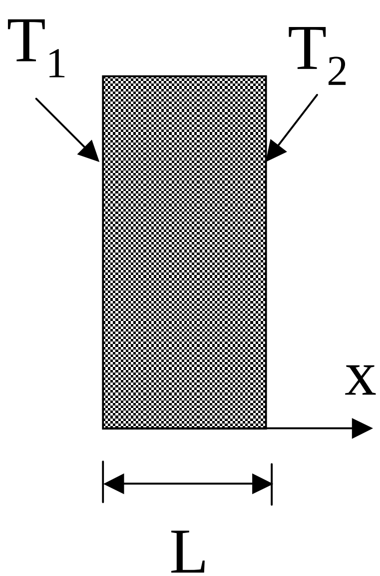
$$q = -kA\frac{dT}{dx}, \quad \text{If the temperature gradient is a constant}$$

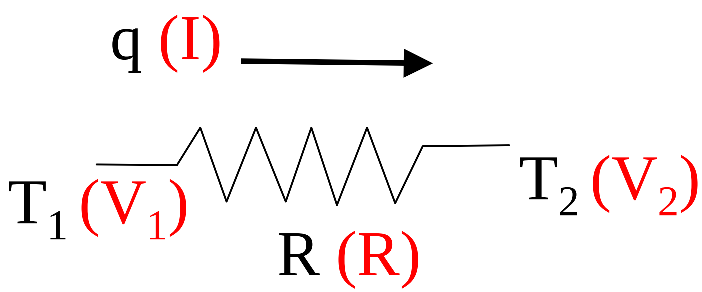
q = constent (heat transfer rate is a constant)

$$\frac{dT}{dx} = \frac{T_2 - T_1}{L}, \text{ where } T(x=0) = T_1, T(x=L) = T_2$$

$$q = -kA\frac{dT}{dx} = kA\frac{T_1 - T_2}{L} = \frac{T_1 - T_2}{(L/kA)}$$

$$q = \frac{T_1 - T_2}{R}, where\ \ R = \frac{L}{kA} : \text{thermal resistance}$$

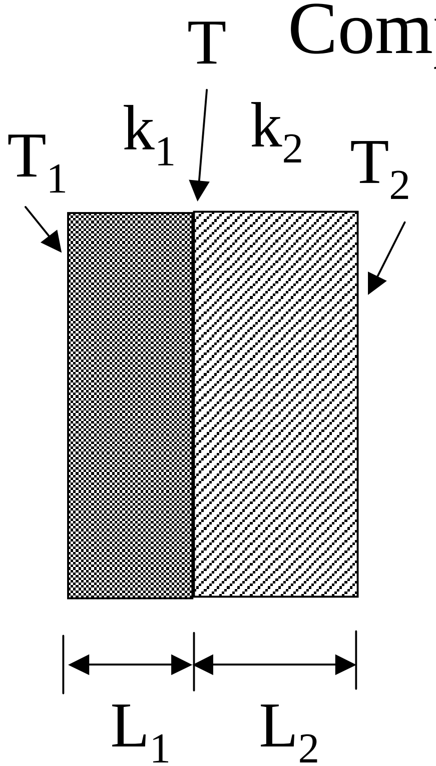




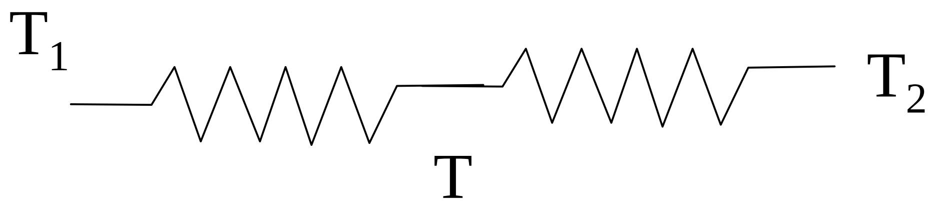
Electric circuit analogy

$$I = (V_1 - V_2)/R$$

Composite Wall Heat Transfer



$R_1 = L_1/(k_1 A)$ $\quad$ $R_2 = L_2/(k_2 A)$



$$q = \frac{T_1 - T_2}{R} = \frac{T_1 - T_2}{R_1 + R_2} = \frac{T_1 - T_2}{\left(\frac{L_1}{k_1 A}\right) + \left(\frac{L_2}{k_2 A}\right)}$$

Also, $q = \frac{T_1 - T}{R_1}$, $T = T_1 - qR_1 = T_1 - q\left(\frac{L_1}{k_1 A}\right)$

Note: In the US, insulation materials are often specified in terms of their thermal resistance in (hr ft^2 °F)/Btu ----> 1 Btu=1055 J.
R-value = L/k, R-11 for wall, R-19 to R-31 for ceiling.

R value

The thermal resistance of insulation material can be characterized by its R-value. R is defined as the temperature difference across the insulation by the heat flux going through it:

$$R = \frac{\Delta T}{q''} = \frac{\Delta T}{k \Delta T / \Delta x} = \frac{\Delta x}{k}$$

The typical space inside the residential frame wall is 3.5 in. Find the R-value if the wall cavity is filled with fiberglass batt. (k=0.046 W/m.K=0.027 Btu/h.ft.R)

$$R = \frac{\Delta x}{k} = \frac{0.292\ ft}{0.027\, Btu/h.\,ft.\,R} = 10.8(R.\,ft^2.h/Btu) \approx R-11$$