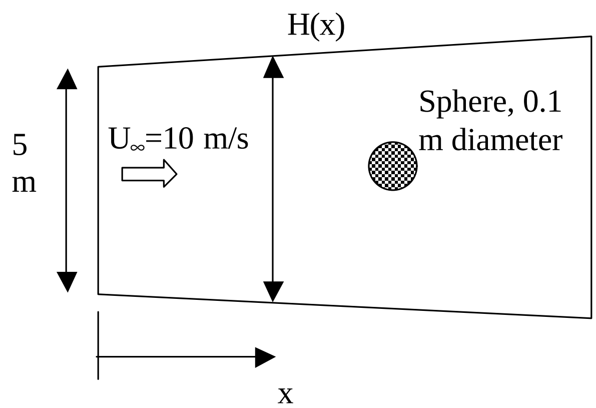
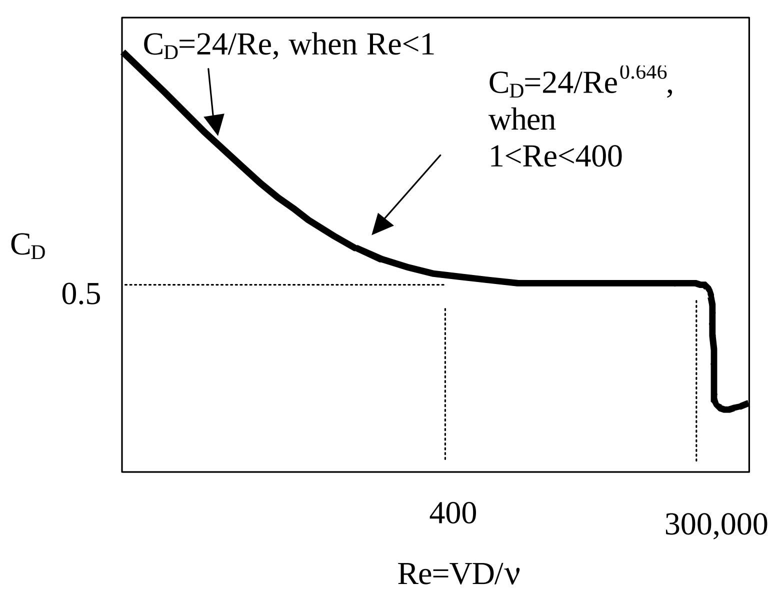
Example: Air enters a two-dimensional wind tunnel as shown. Because of the presence of the boundary layer, fluid will be displaced away from the surface. In order to maintain a constant velocity inside the tunnel, it is necessary to increase the cross-sectional size of the tunnel.

(a) Determine the channel height, H(x), as a function of the distance measured from the inlet of the tunnel, x. The tunnel velocity is 10 m/s and the tunnel inlet height is 5 m. Assume the boundary layer has a profile $u(y)=U_\infty \sin(\pi y/2\delta)$.

(b) What is the momentum thickness $\theta(x)$ of the boundary layer flow.

(c) If a sphere has a diameter of 0.1 m is placed in the center of the wind tunnel, what is the drag force exerted on the sphere. $\rho_{air}=1.2$ kg/m^3, $\nu=1.5\times10^{-5}$ m^2/s.





In order to maintain a constant velocity inside the tunnel, the tunnel wall has to be displaced outward in order to accomodate the growth of the boundary layer. The mass flow displaced by the presence of the boundary layer can be related to $U_\infty d^*$ by the definition of the displacement thickness. Therefore, the tunnel height has to be $H(t) = 5 + 2d^*(x)$.

$$d^* = \int_0^\infty (1 - \frac{u}{U_\infty}) dy = \int_0^d [1 - \sin(\frac{py}{2d})] dy = d + \frac{2d}{p}\cos(\frac{py}{2d})\Big|_0^d = d + (0 - \frac{2d}{p}) = (1 - \frac{2}{p}) d(x) = 0.363 d(x)$$

$$H(x) = 5 + 0.726 d(x)$$

From the momentum thickness equation: $t_w = rU_\infty^2 \frac{dq}{dx}$

where $q = \int_0^\infty \frac{u}{U_\infty}(1 - \frac{u}{U_\infty}) dy$ and $t_w = m\frac{\partial u}{\partial y}|_{y=0}$.

$$t_w = m\frac{\partial}{\partial y}[\sin(\frac{py}{2d})]_{y=0} = \frac{mp}{2d}\cos(\frac{py}{2d})_{y=0} = \frac{mp}{2d}$$

$$q = \int_0^d \sin(\frac{py}{2d})(1 - \sin(\frac{py}{2d})) dy = \int_0^d [\sin(\frac{py}{2d}) - \sin^2(\frac{py}{2d})] dy = -\frac{2d}{p}\cos(\frac{py}{2d})\Big|_0^d - \int_0^d [\frac{1}{2} - \frac{1}{2}\cos(\frac{py}{d})] dy$$

$$= \frac{2d}{p} - \frac{d}{2} - \frac{d}{2p}\sin(\frac{py}{d})\Big|_0^d = \frac{2d}{p} - \frac{d}{2} = (\frac{2}{p} - \frac{1}{2}) d$$

$$\frac{dq}{dx} = (\frac{2}{p} - \frac{1}{2})\frac{dd}{dx} \Rightarrow \frac{mp}{2d} = rU_\infty^2 (\frac{2}{p} - \frac{1}{2})\frac{dd}{dx} \Rightarrow \frac{mp}{rU_\infty^2(\frac{4}{p} - 1)} dx = d\,dd$$

Integrate: $\frac{mp}{rU_\infty^2(\frac{4}{p} - 1)} x = \frac{d^2}{2}$, $d(x) = \sqrt{\frac{2mp^2}{rU_\infty^2(4 - p)}}\sqrt{x} = 1.595 \times 10^{-3}\sqrt{x}\,(m)$

$$d^*(x) = 0.363 d(x) = 5.79 \times 10^{-4}\sqrt{x}\ (m)$$

$(a)\ H(x) = 5 + 2d^*(x) = 5 + 1.158 \times 10^{-3}\sqrt{x}\ (m)$

(b) The momentum thickness $q(x) = (\frac{2}{p} - \frac{1}{2}) d = (\frac{2}{p} - \frac{1}{2})(1.595 \times 10^{-3})\sqrt{x} = 2.18 \times 10^{-4}\sqrt{x}\,(m)$

(c) For the sphere: $\mathrm{Re} = \frac{U_\infty D}{n} = \frac{(10)(0.1)}{1.5 \times 10^{-5}} = 66{,}667, \quad 300 < \mathrm{Re} < 300{,}000 \Rightarrow C_D = 0.5$

$C_D = \frac{F_D}{\frac{1}{2} rU_\infty^2 (\frac{p}{4} D^2)}$, $F_D = (0.5)(1/2)(1.2)(10)^2(p/4)(0.1)^2 = 0.235(N)$ drag force