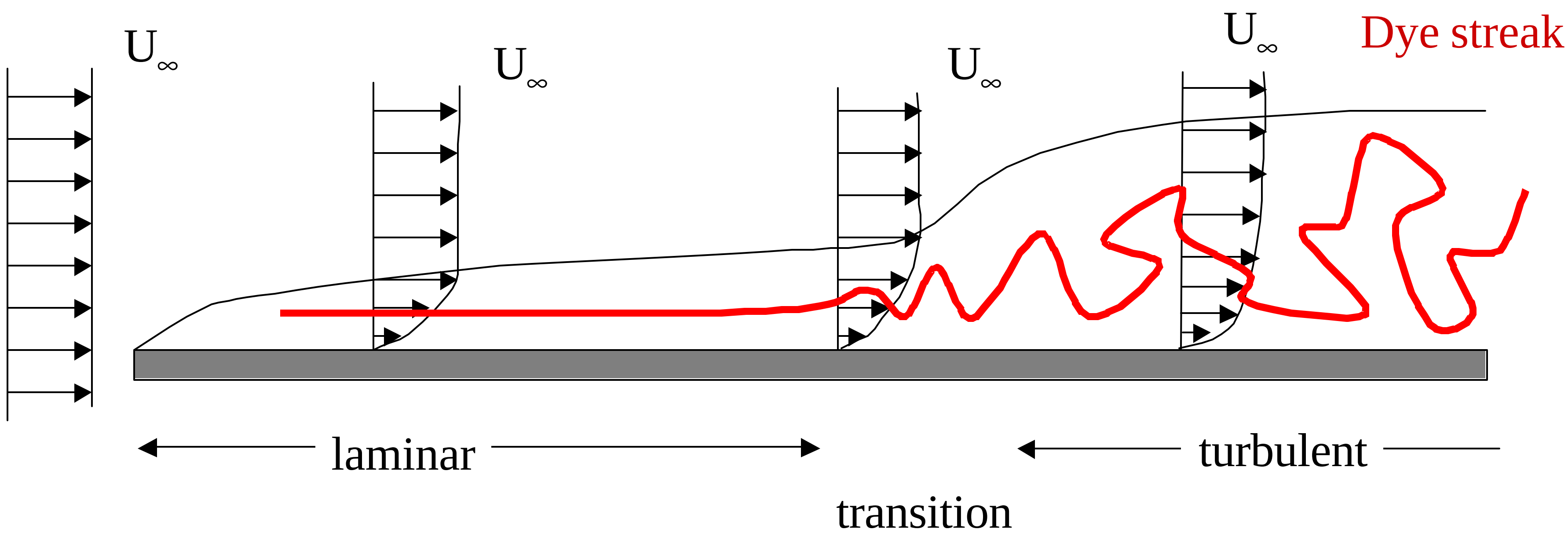
External Flows

An internal flow is surrounded by solid boundaries that can restrict the development of its boundary layer, for example, a pipe flow. An external flow, on the other hand, are flows over bodies immersed in an unbounded fluid so that the flow boundary layer can grow freely in one direction. Examples include the flows over airfoils, ship hulls, turbine blades, etc.

One of the most important concepts in understanding the external flows is the boundary layer development. For simplicity, we are going to analyze a boundary layer flow over a flat plate with no curvature and no external pressure variation.



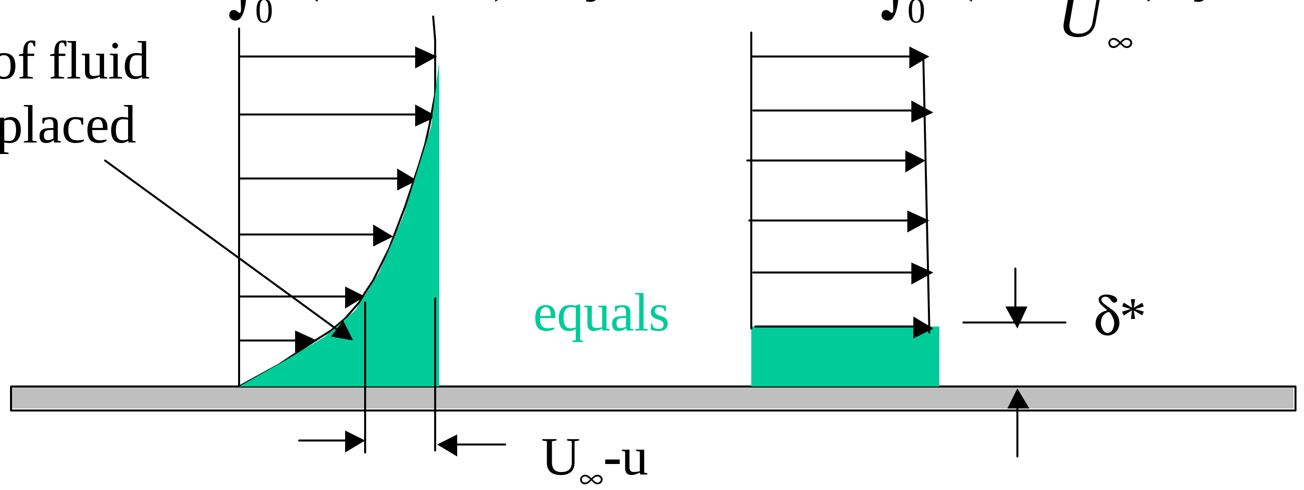
Boundary Layer Definition

➢ Boundary layer thickness (δ): defined as the distance away from the surface where the local velocity reaches to 99% of the free-stream velocity, that is $u(y=\delta)=0.99U_\infty$. Somewhat an easy to understand but arbitrary definition.

➢ Displacement thickness (δ*): Since the viscous force slows down the boundary layer flow, as a result, certain amount of the mass has been displaced (ejected) by the presence of the boundary layer (to satisfy the mass conservation requirement). Image that if we displace the uniform flow away from the solid surface by an amount δ*, such that the flow rate with the uniform velocity will be the same as the flow rate being displaced by the presence of the boundary layer.

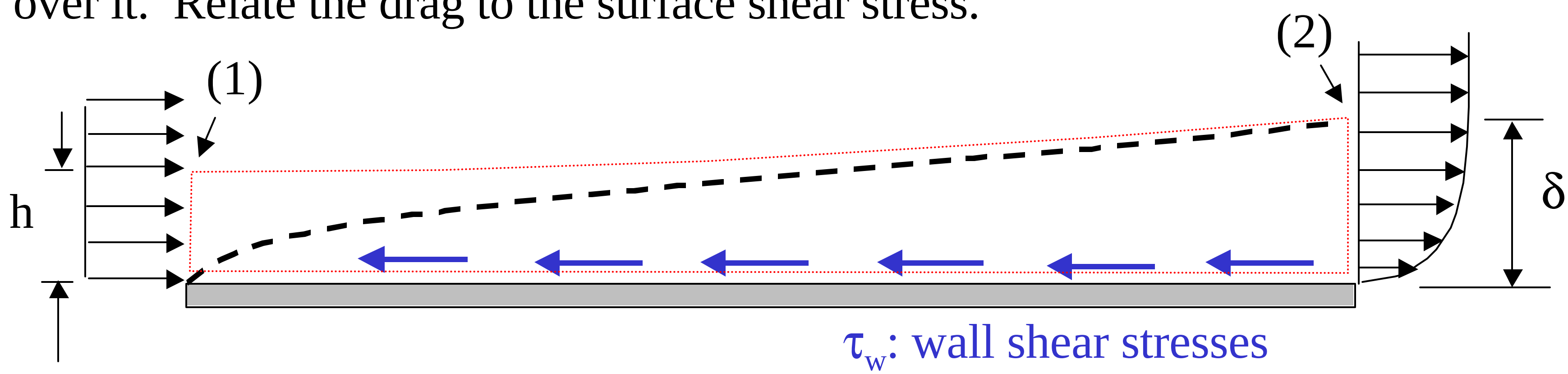
$$d^* U_\infty w = \int_0^\infty (U_\infty - u) w\,dy, \quad \text{or } d^* = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right) dy$$

Amount of fluid being displaced outward



Momentum Balance

Example: Determine the drag force acting on a flat plate when a uniform flow past over it. Relate the drag to the surface shear stress.



Net Force = change of linear momentum

$$F_s = \int_{\text{all surfaces}} \vec{V}(r\vec{V}\cdot d\vec{A}) = \int_{\text{surface(1)}} \vec{V}(r\vec{V}\cdot d\vec{A}) + \int_{\text{surface(2)}} \vec{V}(r\vec{V}\cdot d\vec{A})$$

$$= r\int_{(1)} U_\infty(-U_\infty)dA + r\int_{(2)} u^2\, dA = rU_\infty^2 h - r\int_0^d u^2 dy. \text{ (Assume unit width)}$$

From mass conservation: $U_\infty h = \int_0^d u dy, \quad rU_\infty^2 h = r\int_0^d U_\infty u dy$

$F_s = r\int_0^d u(u - U_\infty)dy.$ Force F_s is the surface acting on the fluid

Skin Friction

The force acting on the plate is called the friction drag (D) (due to the presence of the skin friction).

$$\mathrm{D} = -\mathrm{F_s} = r\int_0^d u(U_\infty - u)dy$$

The drag is related to the deficit of momentum flux across the boundary layer. It can also be directly determined by the integration of the wall shear stress over the entire plate surface:

$$\mathrm{D} = \int_{plate} t_{\mathrm{w}} dA = \int_{plate} t_w dx$$

Define momentum thickness (q) : thickness of a layer of fluid with a uniform velocity U_∞ and its momentum flux is equal to the deficit of boundary layer momentum flux.

$$rU_\infty^2 q = \int_0^\infty r\,u(U_\infty - u)dy, \quad q = \int_0^\infty \frac{u}{U_\infty}(1-\frac{u}{U_\infty})dy$$

Wall Shear Stress and Momentum Thickness

Therfore, the drag force can be related to the momentum thickness as $D = r U_\infty^2 q$, for a unit width boundary layer and this relation is valid for laminar or turbulent flows.

It is also known that $D = \int_{plate} t_w dx$,

$$\frac{dD}{dx} = t_w = r U_\infty^2 \frac{dq}{dx}$$

Shear stress t_w can be directly related to the gradient of

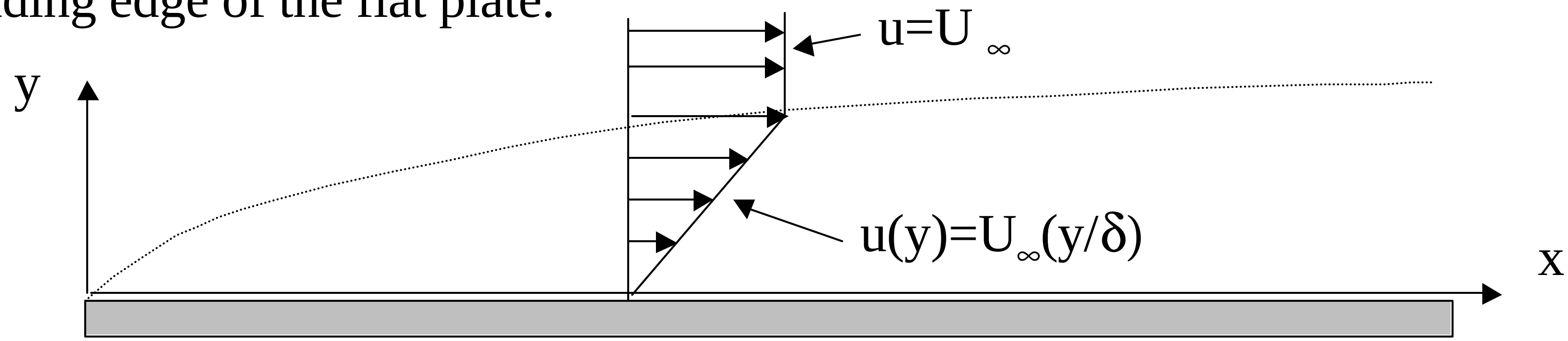
the momentum thickness along the streamwise direction $\frac{dq}{dx}$.

Recall that, for laminar flow, the wall shear stress is defined as:

$$t_w = m(\partial u / \partial y)_{y=0}$$

Example

Assume a laminar boundary layer has a velocity profile as $u(y)=U_\infty(y/\delta)$ for $0\le y\le\delta$ and $u=U_\infty$ for $y>\delta$, as shown. Determine the shear stress and the boundary layer growth as a function of the distance x measured from the leading edge of the flat plate.



$$t_w = r\,U_\infty^2 \frac{dq}{dx}$$

For a laminar flow $t_w = m\left(\partial u / \partial y\right)_{y=0} = m\dfrac{U_\infty}{d}$ from the profile.

Substitute into the definition of the momentum thickness:

$$q = \int_0^\infty \frac{u}{U_\infty}\left(1-\frac{u}{U_\infty}\right)dy = \int_0^d \frac{y}{d}\left(1-\frac{y}{d}\right)dy, \quad \text{since } u = \frac{U_\infty y}{d}$$

$$q = \frac{d}{6}.$$

Example (cont.)

$$t_w = r\mathrm{U}_\infty^2 \frac{dq}{dx}, \quad m\frac{U_\infty}{d} = r\mathrm{U}_\infty^2 \frac{1}{6}\frac{dd}{dx}$$

Separation of variables: $\dfrac{6m}{r\mathrm{U}_\infty} = d\,dd$, integrate $d^2 = \dfrac{12m}{r\mathrm{U}_\infty}x = 12(\dfrac{m}{r\mathrm{U}_\infty x})x^2$,

$$\frac{d}{x} = 3.46\sqrt{\frac{n}{U_\infty x}} = 3.46\frac{1}{\sqrt{\mathrm{Re}_x}}, \text{ where } \mathrm{Re_x} = \frac{U_\infty x}{n}$$

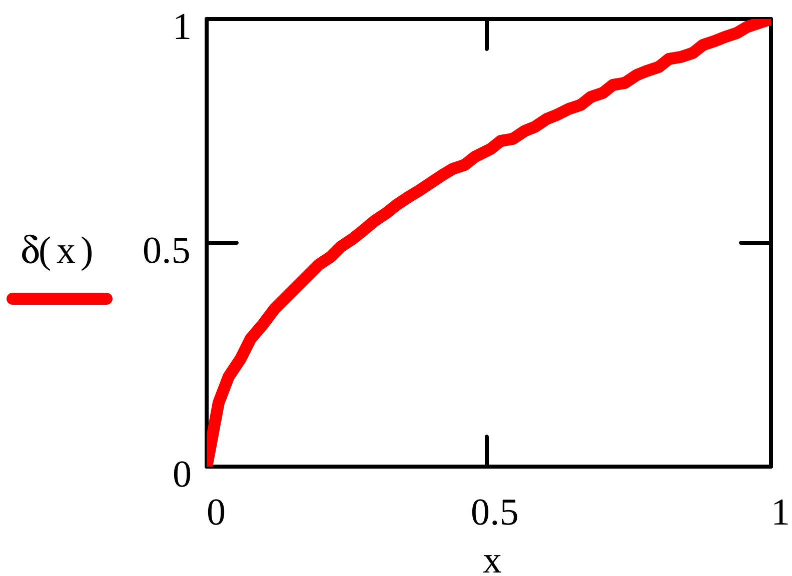
$$d = 3.46\sqrt{\frac{n\,x}{U_\infty}}, \quad d \propto \sqrt{x}$$

$$t_w = m\frac{U_\infty}{d} = 0.289\sqrt{\frac{rmU_\infty^3}{x}} = \frac{0.289\,rU_\infty^2}{\sqrt{\mathrm{Re}_x}}, \quad t_w \propto \frac{1}{\sqrt{x}}$$

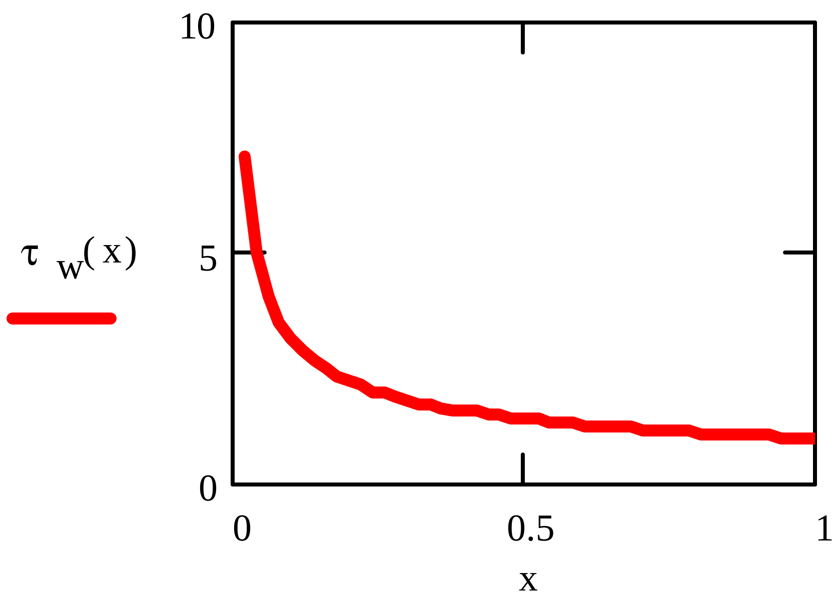
Note: In general, the velocity distribution is not a straight line. A laminar flat-plate boundary layer assumes a Blasius profile (chapter 9.3). The boundary layer thickness δ and the wall shear stress τ_w behave as:

$$d = \frac{5.0}{\sqrt{U_\infty / nx}} = \frac{5.0x}{\sqrt{\mathrm{Re}_x}}, (9.13). \quad t_w = \frac{0.332\,rU_\infty^2}{\sqrt{\mathrm{Re}_x}}, (9.14).$$

Laminar Boundary Layer Development



- Boundary layer growth: $\delta \propto \sqrt{x}$
- Initial growth is fast
- Growth rate $d\delta/dx \propto 1/\sqrt{x}$, decreasing downstream.



- Wall shear stress: $\tau_w \propto 1/\sqrt{x}$
- As the boundary layer grows, the wall shear stress decreases as the velocity gradient at the wall becomes less steep.