Compressible Flow

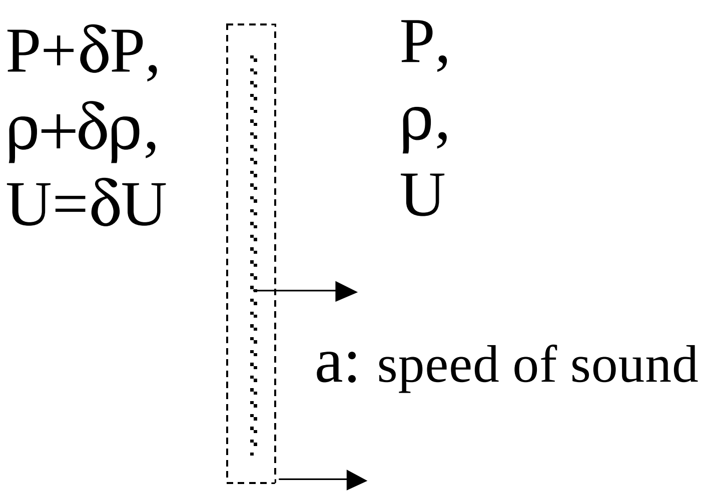
- Density might not be constant when the flow velocity is high. How high? In general, the compressibility effect is proportional to the square of Mach number ~ M^2. Therefore, for the case of M=0.3, it will contribute to roughly 10% error by neglecting compressibility effect.
- Usually, temperature variation can be significant when the flow velocity is high. Consequently, energy equation is important also.
- The relationship between temperature and density can still be characterized by the ideal gas model. (might not be valid for extremely high speed flow)
- In this class, most flow analysis discussed will be considered as one-dimensional for simplicity. However, it is understood that multi-dimensional effect can be significant in real flow configurations.

Mach Number and Speed of Sound

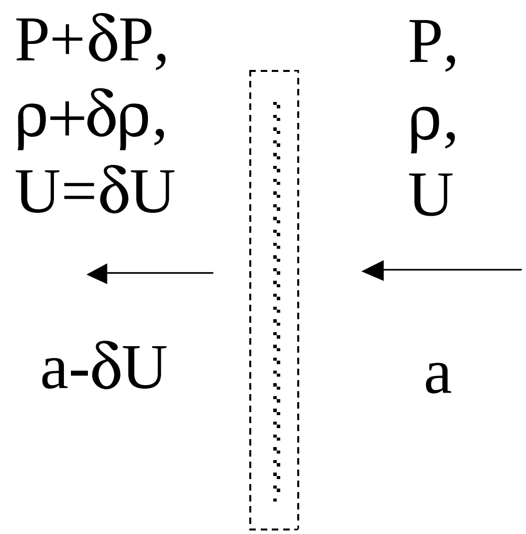
• Mach number (M) is the ratio of the local flow velocity (U) to the local speed of sound (a) : M=U/a.

• Speed of sound (a) is the speed of the sound wave (infinitesimal pressure wave) which travels through the medium (usually air).

• Assume we can follow this wave and observe the variation of the flow field as shown below:



Moving reference frame travelling at the speed of sound



Relative to the moving reference frame

Speed of Sound

From mass conservation:

Negligible. Why?

$$r Aa = (r + dr)A(a - dU)$$

$$r a = r a - r dU + a dr - (dr)(dU) \Rightarrow r dU = a dr$$

From momentum equation:

$$-PA + (P + dP)A = \dot{m}a - (\dot{m} + d\dot{m})(a - dU)$$

$$A dP = (r a^2 A) - (r + dr)(a - dU)^2 A$$

$$r dU = \frac{dP}{a} \text{ (all higher order terms have been neglected)}$$

Therefore, $a^2 = \dfrac{dP}{dr}$, or $a = \sqrt{\dfrac{dP}{dr}}$

Speed of sound (cont.)

• Because sound waves can be considered weak pressure waves, it is reasonable to assume the process is isentropic and the pressure and density variations are both small ($\delta P \rightarrow 0$, $\delta \rho \rightarrow 0$). Therefore, the definition of the speed of sound is given as

$$a = \sqrt{\left(\frac{\partial P}{\partial r}\right)_S},$$

where the subscript s represents an isentropic process.

It can be easily shown that, for an ideal gas, the speed of sound can be further simplified as

$$a = \sqrt{g\,RT}$$

Ex: the speed of sound of air at 0°C is about 330 m/s, very high!

Isentropic Deceleration/Acceleration Process

- Assume 1-D flow, compressible, adiabatic, reversible (isentropic process)

If the flow is incompressible, we can integrate the momentum equation (Euler's equ) to obtain the famous Bernoulli equation:

$$\frac{\mathrm{P}}{r} + \frac{U^2}{2} = \text{constant (assume negligible potential energy variation)}$$

For a compressible flow, a similar expression (before integrating the momentum equation) can be obtained

(see pp 602, chapter 11 IFM): $\frac{dP}{r} + d\left(\frac{U^2}{2}\right) = 0$

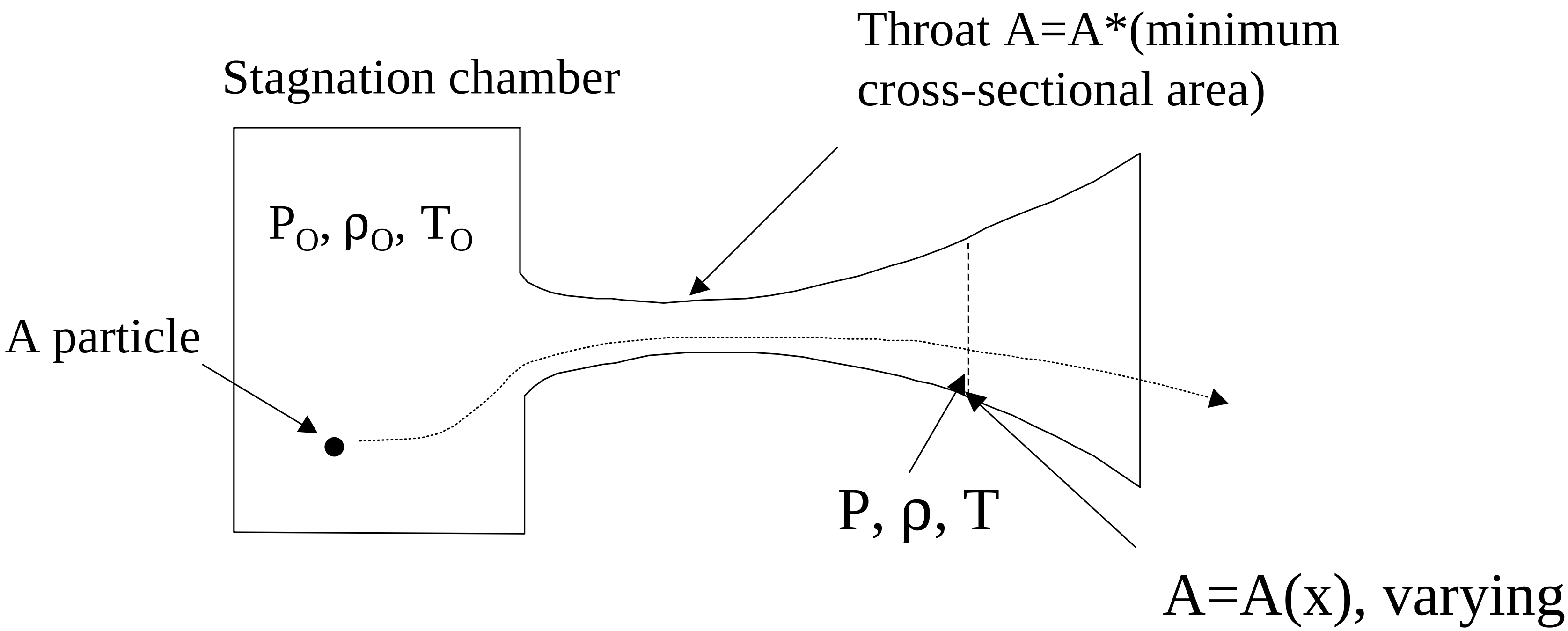
Isentropic Deceleration/Acceleration Process (cont.)

If we integrate the previous equation for an incompressible flow (r =constant), then we can obtain the Bernoulli equation. However, if the density is not a constant, a relation between the density and the pressure has to be specified. using the ideal gas assumption and isentropic process analysis, they can be related as

$$\frac{P}{r^{g}} = \text{constant}$$

Now, we can integrate the momentum equation from its stagnation condition to any downstram location using the operating configuration defined in the next page. (Image, a fluid particle is accelerating through a channel as a result of the pressure difference between the stagnation reservoir and the ambient).

Isentropic Deceleration/Acceleration Process



It can be shown that the relation between the stagnation pressure (P_O) and pressure at any section (P) is given as:

$$\frac{P_O}{P} = \left[1 + \frac{g-1}{2} M^2\right]^{g/(g-1)}$$, where M is the local Mach number.

Isentropic Deceleration/Acceleration Process

It can also be shown that (using ideal gas assumption and isentropic relation)

$$\frac{T_O}{T} = \left[1 + \frac{g - 1}{2} M^2\right]$$

$$\frac{r_O}{r} = \left[1 + \frac{g - 1}{2} M^2\right]^{1/(g-1)}$$

Show that the same relations can be obtained using the following energy equation:

$$h_O = h + \frac{U^2}{2}, \text{ or } c_p T_O = c_p T + \frac{U^2}{2}$$