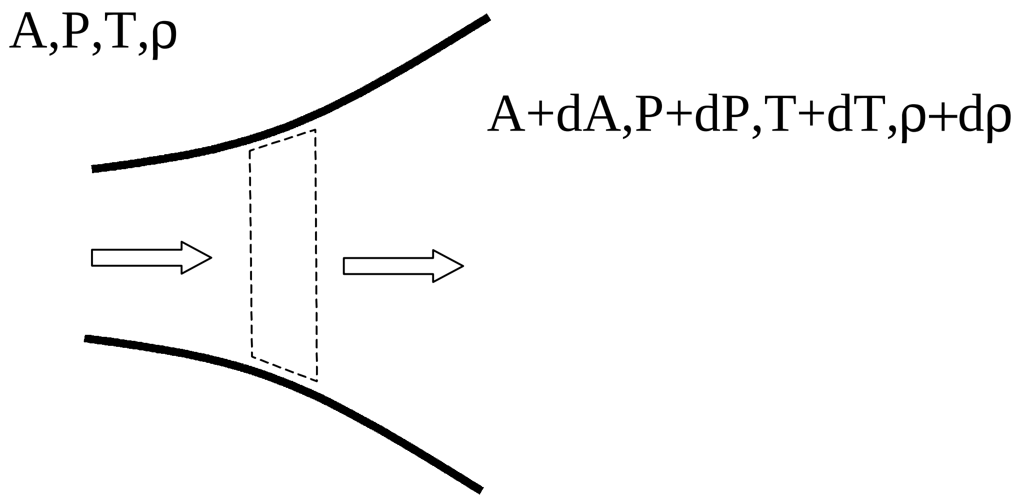
Variable Area Flow



From mass conservation: ρAV=($\rho + d\rho$)(A+dA)(U+dU)

By neglecting all higher order terms (such as (dρ)(dA), etc..),

we can obtain: ρAdU+ρUdA+AUdρ=0.

Divide all terms by ρAU: $\dfrac{\mathrm{d}\rho}{\rho} + \dfrac{dA}{A} + \dfrac{dU}{U} = 0$

Density/Velocity Variation

$\Rightarrow$ If the flow is incompressible (ρ=constant):

$\frac{\mathrm{dU}}{U} = -\frac{dA}{A}, dA > 0 \Rightarrow dU < 0$ and vice versa as predicted by the Bernoulii's equation

$\Rightarrow$ If the flow is compressible and d$\rho \neq 0$, the density variation can play a significant role in determined the velocity variation in addition to the area change

It can be shown that (see p 618, chapter 12-2, IFM): $\frac{dU}{U} = \frac{1}{(M^2 - 1)} \frac{dA}{A}$

Clearly, the area and velocity relationship changes and it is now a function of the local Mach number (M) also.

Density/Velocity Variation

This change is due to the fact that the variation of density changes drastically w.r.t the velocity change. It can be shown that:

$$\frac{d\rho}{\rho} = -M^2 \frac{dU}{U}$$

From this equation, it can be clearly seen that, density decrease with an increase of velocity and vice versa (Explain Why?).

Interestingly, the change is different for a subsonic flow (M<1) as compared to a supersonic flow (M>1). One can say that density variation ($d\rho/\rho$) is more drastic than velocity variation (dU/U) under this circumstance since $\left|\frac{d\rho}{\rho}\right| > \left|\frac{dU}{U}\right|$. Due to this transition, we can expect that the flow behavior will change when M=1 (that is the local sonic condition).

Velocity/Area Variation

We have drived earlier that the velocity/area relation

of a variable area flow is given by: $\frac{dU}{U} = \frac{1}{(M^2 - 1)} \frac{dA}{A}$

$\Rightarrow$ If M<1 (subsonic flow): decreasing the area leads to the increase of velocity (flow accelerates). UNTIL IT APPROACHES M=1 !!

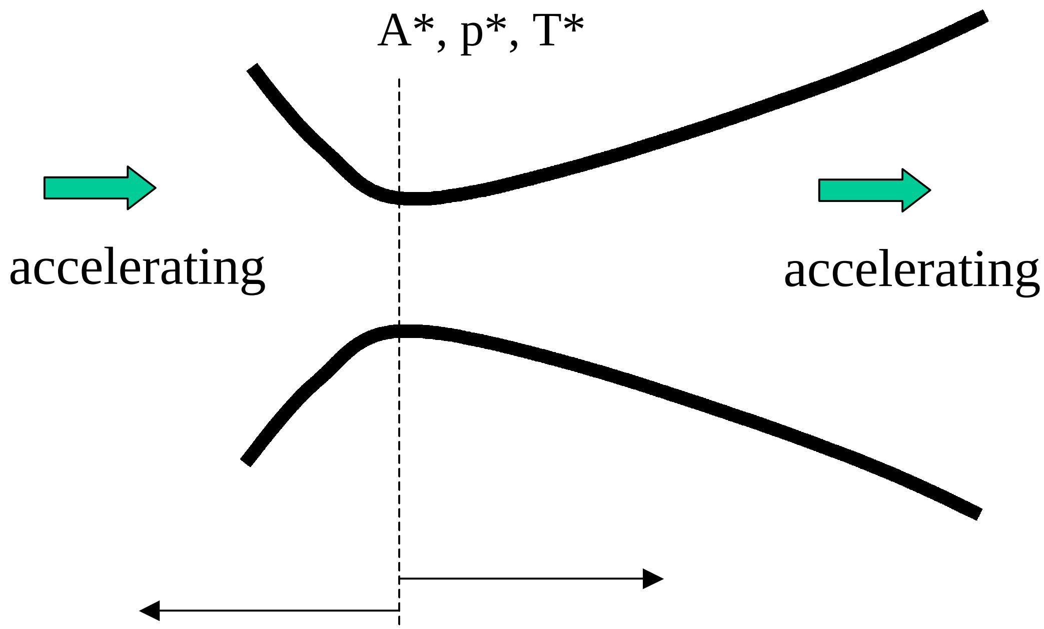
$\Rightarrow$ If M>1 (supersonic flow): decreasing the area decreases the flow velocity (flow decelerates ??). FLOW CAN NOT EXCEED SPEED OF SOUND THROUGH DECREASING AREA ANYMORE. Therefore, the area needs to be increased if further flow acceleration is necessary.

$\Rightarrow$ If M=1, $\frac{\mathrm{dA}}{\mathrm{A}} = (M^2 - 1)\frac{dU}{U} = 0$ Local area must be a maximum or a minimum. From previous discussion, it has to be a minimum (a throat).

Generation of Supersonic Flow in a variable area duct

Sonic flow: M=1 dA=0

Specify local throat condition using *



Subsonic flow
M<1, dA<0,
dU>0, dM>0

Supersonic flow
M>1, dA>0,
dU>0, dM>0

At the throat, the following relations are true:

$$\frac{P_O}{P^*} = \left[\frac{\gamma+1}{2}\right]^{\gamma/(\gamma-1)}$$

$$\frac{T_O}{T^*} = \left[\frac{\gamma+1}{2}\right]$$

$$\frac{\rho_O}{\rho^*} = \left[\frac{\gamma+1}{2}\right]^{1/(\gamma-1)}$$