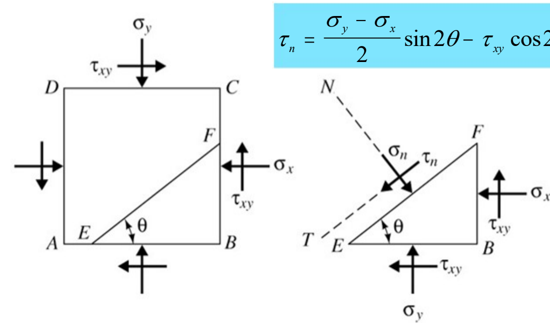
STRESSES BENEATH LOADED AREAS

Stress Transformation

$$\sigma_n = \frac{\sigma_y + \sigma_x}{2} + \frac{\sigma_y - \sigma_x}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_n = \frac{\sigma_y - \sigma_x}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$



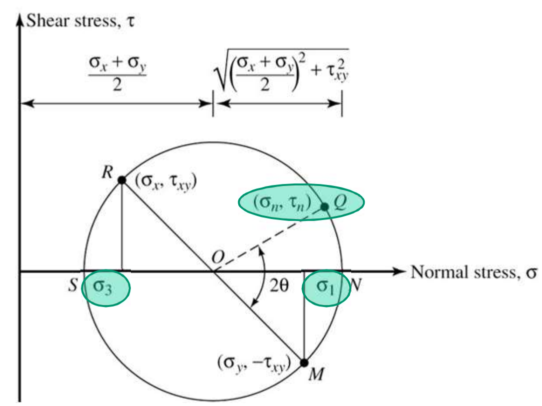
Principal Stresses

- Principal planes – planes for which the normal stresses take extreme values, and shear stresses are zero. Stresses in principal planes are principal stresses.
- Major and minor principal stresses

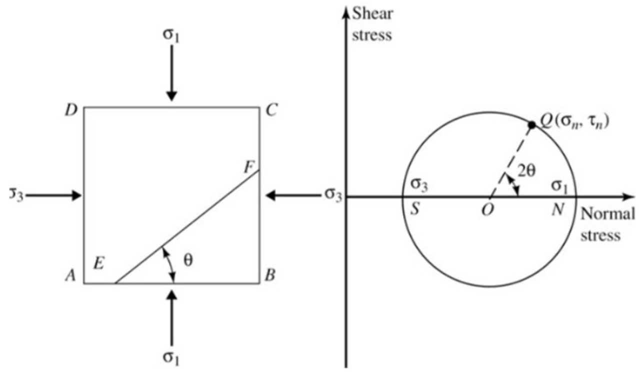
$$\sigma_1 = \frac{\sigma_y + \sigma_x}{2} + \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_y + \sigma_x}{2} - \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

Mohr's Circle (from Das, 2002)



Mohr's Circle for Known Principle Stresses

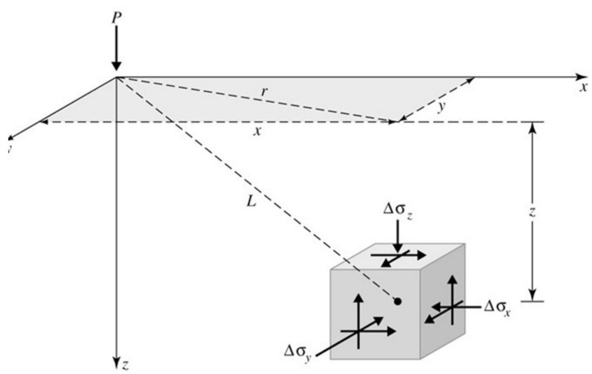


Maximum Shear Stress

- Corresponds to the radius of the Mohr's circle

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

Stresses Due to a Point Load - Boussinesq Solution (1885)

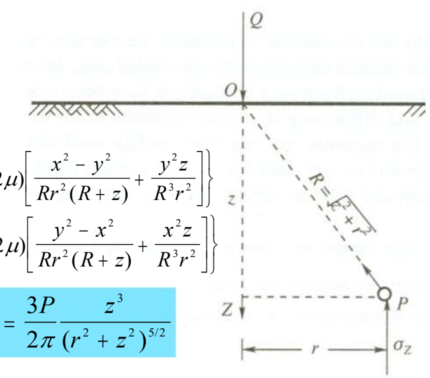


Stresses Due to a Point Load - Boussinesq Solution (1885)

$$\Delta\sigma_x = \frac{P}{2\pi}\left\{\frac{3x^2 z}{R^5} - (1-2\mu)\left[\frac{x^2 - y^2}{Rr^2(R+z)} + \frac{y^2 z}{R^3 r^2}\right]\right\}$$

$$\Delta\sigma_y = \frac{P}{2\pi}\left\{\frac{3y^2 z}{R^5} - (1-2\mu)\left[\frac{y^2 - x^2}{Rr^2(R+z)} + \frac{x^2 z}{R^3 r^2}\right]\right\}$$

$$\Delta\sigma_z = \frac{3P}{2\pi}\frac{z^3}{R^5} = \frac{3P}{2\pi}\frac{z^3}{(r^2 + z^2)^{5/2}}$$



Stresses Due to a Point Load - Boussinesq Solution (1885)

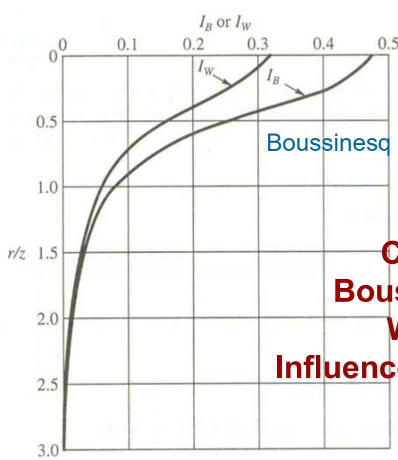
- Alternative formulation

$$\Delta\sigma_z = \frac{P}{z^2}\left\{\frac{3}{2\pi}\frac{1}{\left[\left(\frac{r}{z}\right)^2+1\right]^{5/2}}\right\} = \frac{P}{z^2}I_B$$

Stresses Due to a Point Load – Weestergard's Solution

- Assumes presence of layers of thin, infinitely rigid horizontal reinforcement between layers.
- Increase in pressure

$$\Delta\sigma_z = \frac{Q}{\pi z^2}\frac{1}{\left[2\left(\frac{r}{z}\right)^2+1\right]^{3/2}} = \frac{P}{z^2}I_W$$



Boussinesq more conservative!

Comparison of Boussinesq's and Weestergard's Influence Coefficients

(from Murthy, 2003)

Stresses by Boussinesq and Weestergard – Example (Murthy 6.2)

A concentrated load of 45000 lb acts at foundation level at a depth of 6.56 ft below ground surface. Find the vertical stress along the axis of the load at a depth of 32.8 ft and at a radial distance of 16.4 ft at the same depth by (a) Boussinesq, and (b) Westergaard formulae for $\mu = 0$. Neglect the depth of the foundation.

Solution

(a) Boussinesq Eq. (6.1a)

$$\sigma_z = \frac{Q}{z^2}I_B, \quad I_B = \frac{3}{2\pi}\left[\frac{1}{1+(r/z)^2}\right]^{5/2}$$

Substituting the known values, and simplifying

$I_B = 0.2733$ for $r/z = 0.5$

$$\sigma_z = \frac{45000}{(32.8)^2}\times 0.2733 = 11.43\ \text{lb/ft}^2$$

Stresses by Boussinesq and Weestergard – Example (Murthy 6.2)

(b) Westergaard (Eq. 6.3)

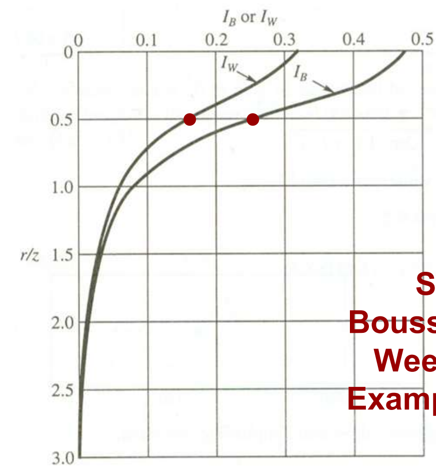
$$\sigma_z = \frac{Q}{z^2} I_w, \quad I_w = \frac{1}{\pi}\left[\frac{1}{1+2(r/z)^2}\right]^{3/2}$$

Substituting the known values and simplifying, we have,

$I_w = 0.1733$ for $r/z = 0.5$

therefore,

$$\sigma_z = \frac{45000}{(32.8)^2} \times 0.1733 = 7.25 \text{ lb/ft}^2$$

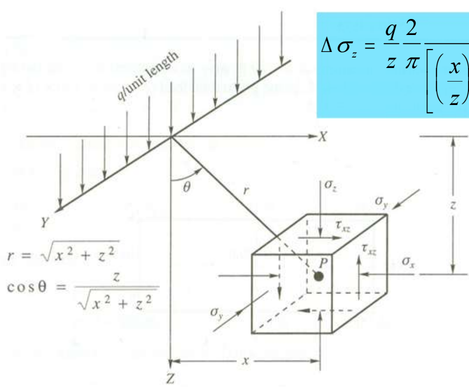


Stresses by Boussinesq and Weestergard – Example (Murthy 6.2)

Vertical Stress Due to a Line Load

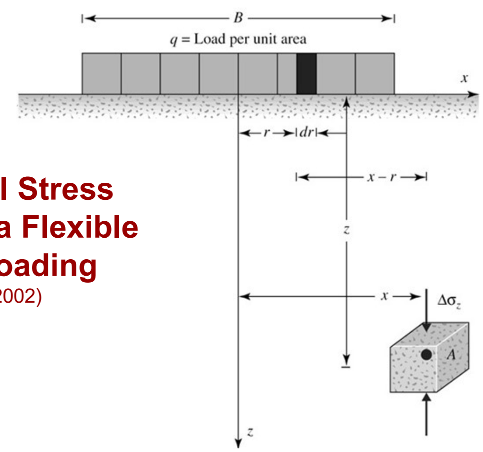
(from Murthy, 2002)

$$\Delta\sigma_z = \frac{q}{z}\frac{2}{\pi}\frac{1}{\left[\left(\frac{x}{z}\right)^2+1\right]^2} = \frac{q}{z} I_z$$



Vertical Stress Under a Flexible Strip Loading

(from Das, 2002)



Vertical Stress Under a Flexible Strip Loading

- From the solution for a line loading, the vertical stress at P for a line load qdx

$$d\sigma_z = \frac{2q}{\pi}\frac{z^3}{[(x-\bar{x})^2+z^2]^2}$$

- The total stress increase at point P is

$$\sigma_z = \frac{2q}{\pi}\int_{-b}^{b}\frac{z^3}{[(x-\bar{x})^2+z^2]^2}dx$$

Vertical Stress Under a Flexible Strip Loading

- Solving the integral

$$\sigma_z = \frac{2q}{\pi}\left[\tan^{-1}\frac{z}{x-b}\tan^{-1}\frac{z}{x+b} - \frac{2bz(x^2-b^2-z^2)}{(x^2-b^2+z^2)+4b^2z^2}\right]$$

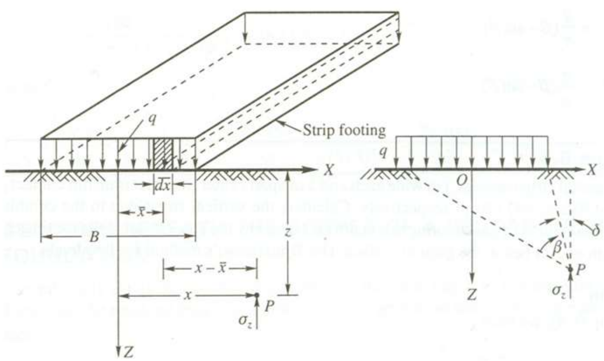
or

$$\sigma_z = \frac{q}{\pi}[\beta + \sin\beta\cos(\beta+2\delta)]$$

- Principal stresses at P are

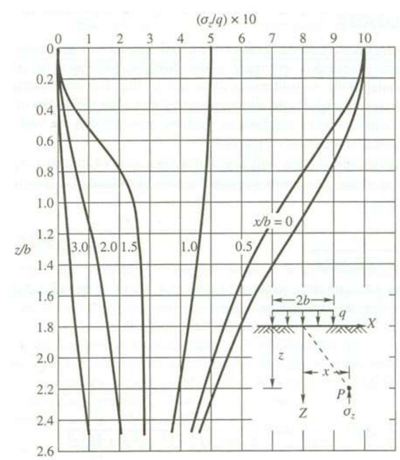
$$\sigma_1 = \frac{q}{\pi}[\beta + \sin\beta] \qquad \sigma_3 = \frac{q}{\pi}[\beta - \sin\beta]$$

Vertical Stress Under a Flexible Strip Loading (from Murthy, 2002)



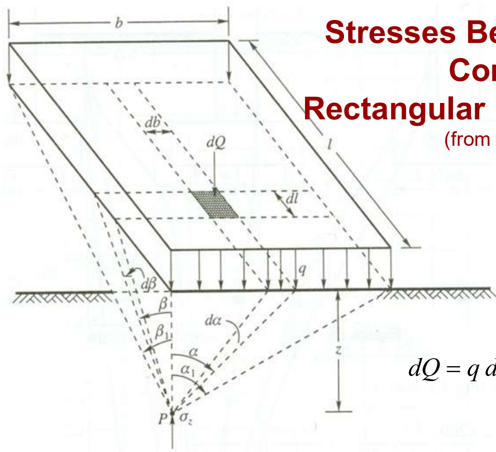
Pressure Ratio for Strip Loading

(from Murthy, 2003)



Stresses Beneath a Corner of a Rectangular Loading

(from Murthy, 2003)



$$dQ = q\,db\,dl$$

Stresses Beneath a Corner of a Rectangular Loading

- Stress increase at P due to dQ

$$d\sigma_z = \frac{dQ}{2\pi}\frac{3z^3}{\left(z^2+r^2\right)^{5/2}}$$

- Integrating in x and y directions from 0 to b, and 0 to l, respectively, or

$$\sigma_z = \int_{\alpha=0}^{\alpha=\alpha_1}\int_{\beta=0}^{\beta=\beta_1} d\sigma_z$$

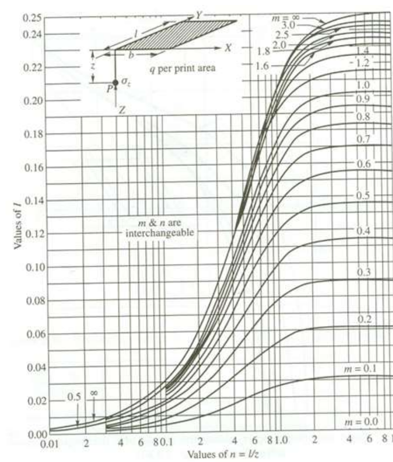
Stresses Beneath a Corner of a Rectangular Loading

- The stress increase at P is

$$\sigma_z = q\frac{1}{4\pi}\left[\frac{2mn(m^2+n^2+1)^{1/2}}{m^2+n^2+m^2n^2+1}\left(\frac{m^2+n^2+2}{m^2+n^2+1}\right)+\tan^{-1}\frac{2mn(m^2+n^2+1)^{1/2}}{m^2+n^2-m^2n^2+1}\right]$$

The obtained result is in radians !!!!

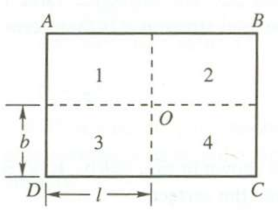
where m=b/z and n=l/z, or $\sigma_z = qI$

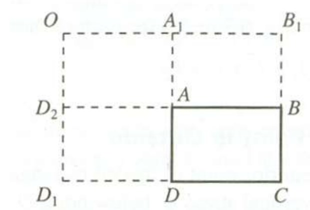


Influence Coefficient for Rectangular Loading

(after Fadum, 1934)

Computation of Stresses Below a Rectangular Loading (from Murthy, 2003)





Point 0 within the area

$\sigma_z = q(I_1 + I_2 + I_3 + I_4)$

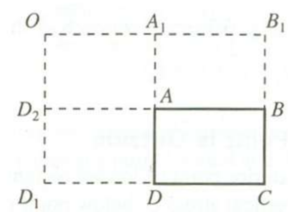
Point 0 outside the area

$\sigma_z = q(I_1 - I_2 - I_3 + I_4)$

Rectangular Loading – Example

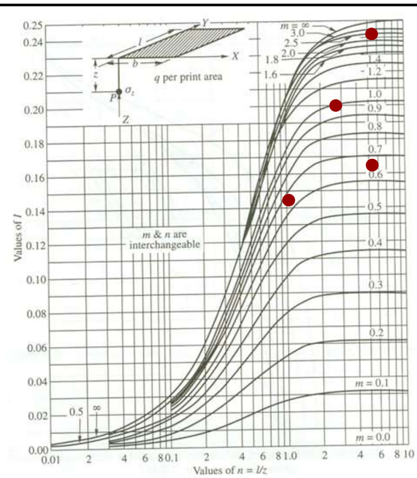
(Murthy 6.5)

ABCD is a raft foundation of a multi-story building [Fig. 6. 9(b)] wherein AB = 65.6 ft, and BC = 39.6 ft. The uniformly distributed load q over the raft is 7310 lb/ft^2. Determine σ_z at a depth of 19.7 ft below point O [Fig. 6.9(b)] wherein AA_1 = 13.12 ft and A_1O = 19.68 ft. Use Fig. 6.8.



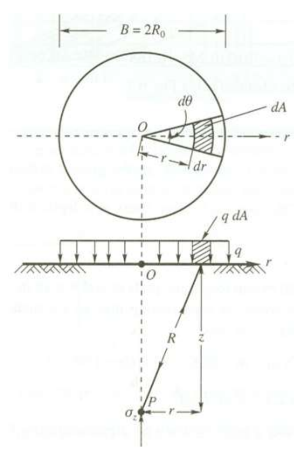
Point 0 outside the area

$\sigma_z = q(I_1 - I_2 - I_3 + I_4)$



Rectangular Loading – Example

(Murthy 6.5)



Stresses Beneath Circular Loads

(from Murthy, 2003)

$$dQ = q\,dA = qr\,d\theta\,dr$$

$$d\sigma_z = \frac{3q}{2\pi}\frac{z^3 r\,d\theta\,dr}{(r^2+z^2)^{5/2}}$$

Stresses Beneath a Circular Loading

- Integrating over the circular area

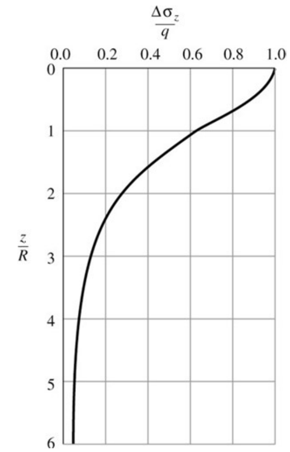
$$\sigma_z = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=R_0} d\sigma_z = \frac{3qz^3}{2\pi}\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=R_0} \frac{r\,d\theta dr}{(r^2+z^2)^{5/2}}$$

the stress at P under the center is

$$\sigma_z = q\left[1 - \frac{z^3}{(R_0^2+z^2)^{3/2}}\right]$$ or $$\frac{\sigma_z}{q} = 1 - \left[\frac{1}{1+\frac{R_0^2}{z^2}}\right]^{3/2} = I_z$$

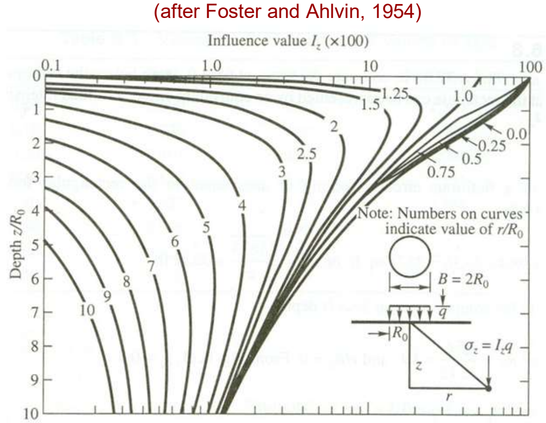
Stresses Beneath the Center of Circular Loading

(from Das, 2002)

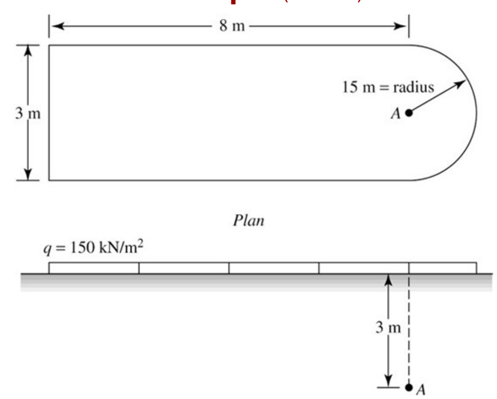


Influence Value Chart for Circular Loading

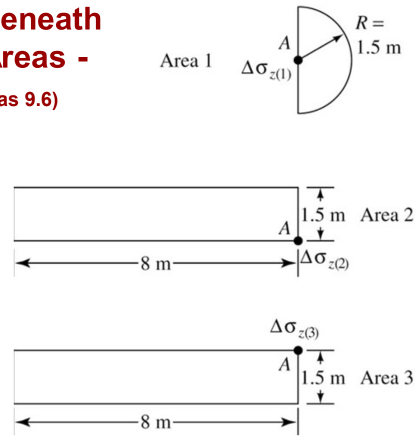
(after Foster and Ahlvin, 1954)



Stresses Beneath Complex Areas - Example (Das 9.6)



Stresses Beneath Complex Areas - Example (Das 9.6)



Influence (Newmark) Chart for General Loading

The solution for a circular loading

$$\sigma_z = q\left[1 - \frac{z^3}{(R_0^2 + z^2)^{3/2}}\right]$$

can be rearranged to

$$\frac{R_0}{z} = \sqrt{\left(1 - \frac{\sigma_z}{q}\right)^{-2/3} - 1}$$

Influence (Newmark) Chart for General Loading

- R_0/z and σ_z/q are dimensionless ratios

  => If geometry is presented in a dimensionless form, for a constant R_0/z, σ_z/q is a constant too.

- Newmark (1942) presented an influence chart for general uniform loading.

Stress Increase for Unit Loading q as a Function of z/R_0 Ratio

(from Das, 2002)

z/R	$\Delta\sigma_z$
0	1
0.02	0.9999
0.05	0.9998
0.10	0.9990
0.2	0.9925
0.4	0.9488
0.5	0.9106
0.8	0.7562
1.0	0.6465
1.5	0.4240
2.0	0.2845
2.5	0.1996
3.0	0.1436
4.0	0.0869
5.0	0.0571

Influence (Newmark) Chart for General Loading

Construction of charts:

1. A set of concentric circles are constructed for σ_z/q =0, 0.1, 0.2, …., 1 (for σ_z/q =0, R_0/z=0; for σ_z/q =1, R_0/z=infinity).
2. The circles are divided by several equally spaced radial lines.
3. The number of created elements in the chart is N, or the influence value of each element becomes 1/N.

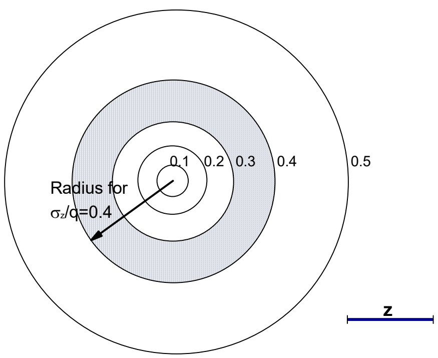
R_0/z Ratio as a Function of Stress Increase Ratio (from Murthy, 2003)

σ_z/q	R/z	σ_z/q	R/z
0.00	0.000	0.80	1.387
0.10	0.270	0.90	1.908
0.20	0.401	0.92	2.094
0.30	0.518	0.94	2.351
0.40	0.637	0.96	2.748
0.50	0.766	0.98	3.546
0.60	0.917	1.00	∞
0.70	1.110	–	–

Influence (Newmark) Chart for General Loading

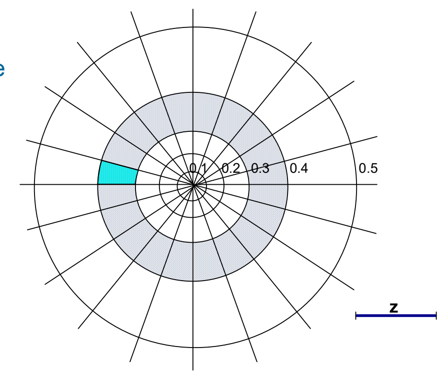
Explanation:

Loading between rings 0.3 and 0.4 causes σ_z/q=0.1.



Influence (Newmark) Chart for General Loading

Loading on the shaded area between rings 0.3 and 0.4 (20 radial lines) causes σ_z/q=0.1/20 =0.005.



Evaluation of Vertical Pressure by Newmark Chart

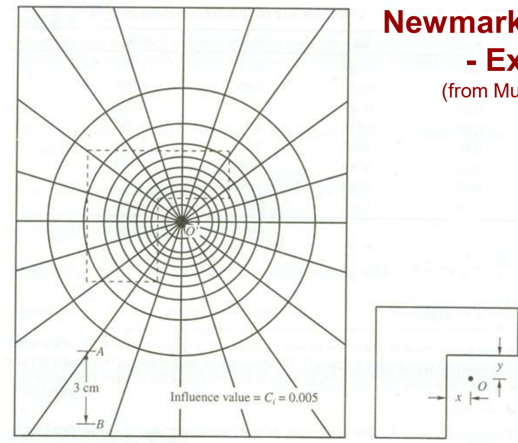
1. Determine the depth (z) of the point of interest relative to the foundation.
2. Plot the plan of the loaded area in a scale where z equals to the unit length of the chart (line AB).
3. Place the plan of the foundation so that the point is located at the center of the chart.
4. Count the number of the elements (N) of the chart covered by the loaded area. The increase in pressure is

$$\sigma_z = C_i N q$$

where C_i is the influence value.

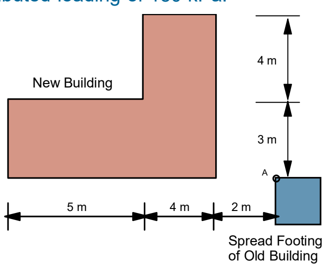
Newmark Chart - Example

(from Murthy, 2003)

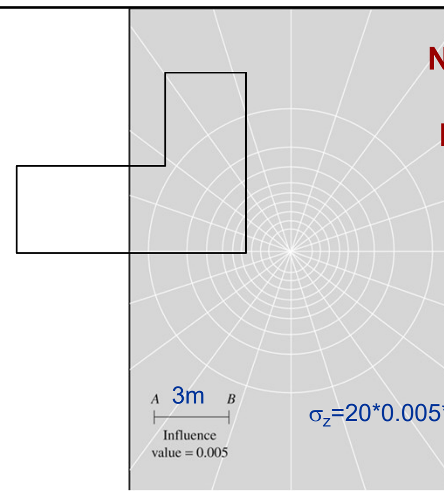


Newmark Chart - Example

A new building is going to be constructed next to an existing one. The bottom of the slab of the new building is 3 m above the bottom of the closest spread footing of the existing building. Find the maximum increase of the vertical pressure beneath the spread footing for the assumption that the new building slab will have a uniformly distributed loading of 150 kPa.

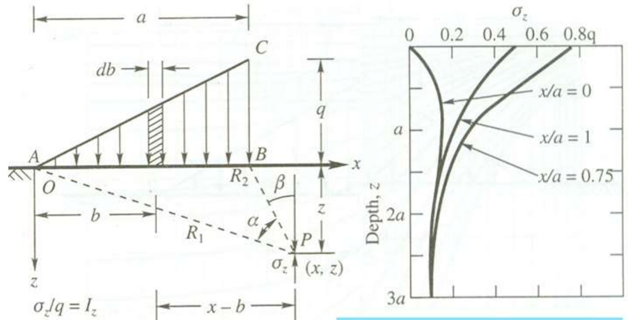


Newmark Chart - Example



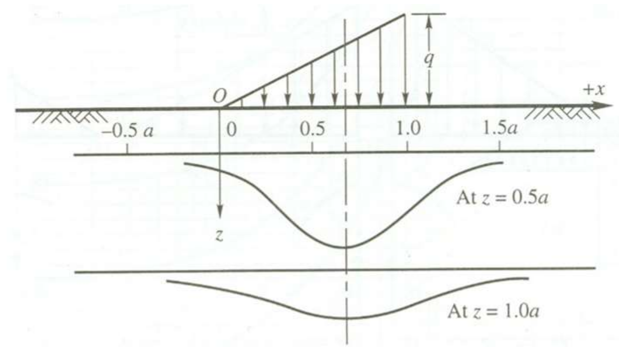
σ_z=20*0.005*150=15 kPa

Triangular Loading (from Murthy, 2003)



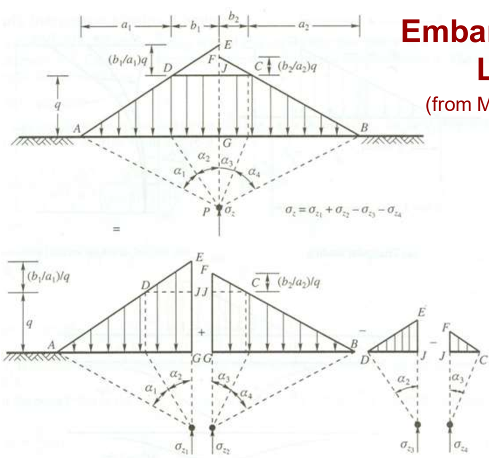
$$\sigma_z = \frac{q}{2\pi}\left(\frac{2xa}{\alpha} - \sin 2\beta\right) = qI_z$$

Triangular Loading (from Murthy, 2003)



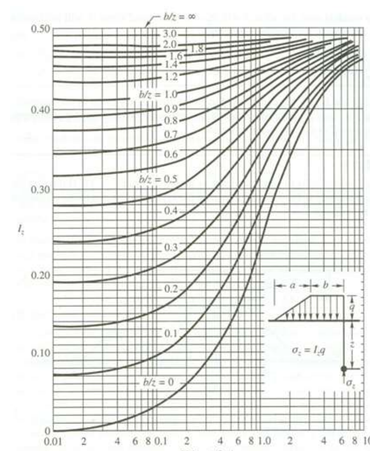
Embankment Loading

(from Murthy, 2003)



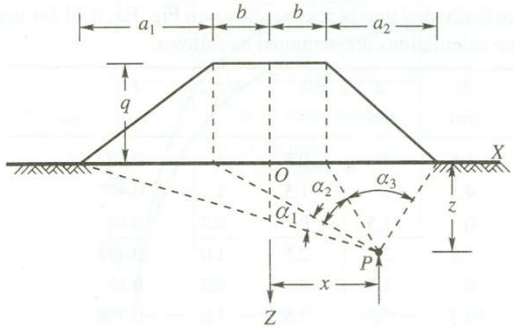
Influence Coefficient for Embankment Loading

(after Osterberg, 1957)



$$\sigma_z = qI_z = \frac{q}{\pi} f(a/z, b/z)$$

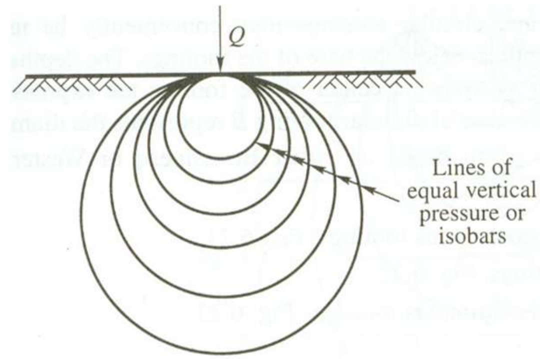
Embankment Loading (from Murthy, 2003)



$$\sigma_z = \frac{q}{\pi}\left[(\alpha_1+\alpha_2+\alpha_3)+\frac{b}{a_1}(\alpha_1+R\alpha_3)+\frac{x}{a_1}(\alpha_1-R\alpha_3)\right]$$

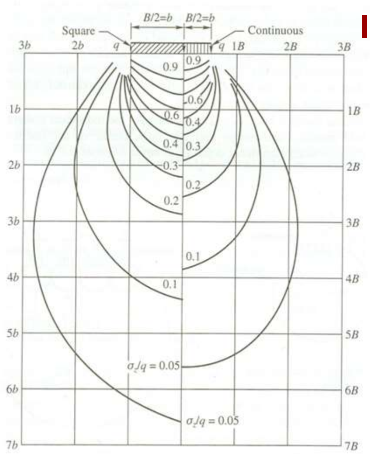
Pressure Isobars

(from Murthy, 2003)



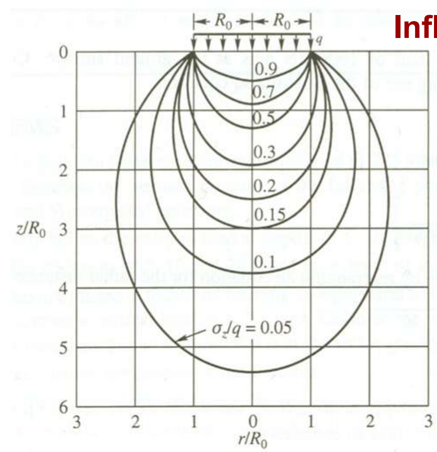
Influence Charts for Square and Strip Loadings

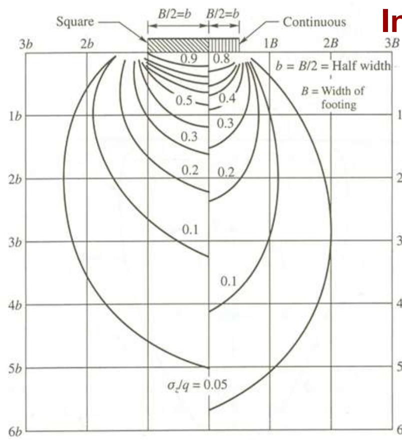
(from Murthy, 2003)



Influence Chart for Circular Loading

(from Murthy, 2003)





Influence Charts for Square and Strip Loads - Westeergard

(from Murthy, 2003)

Pressure Isobars – Example (Murthy 6.12)

A single concentrated load of 1000 kN acts at the ground surface. Construct an isobar for $\sigma_z = 40$ kN/m^2 by making use of the Boussinesq equation.

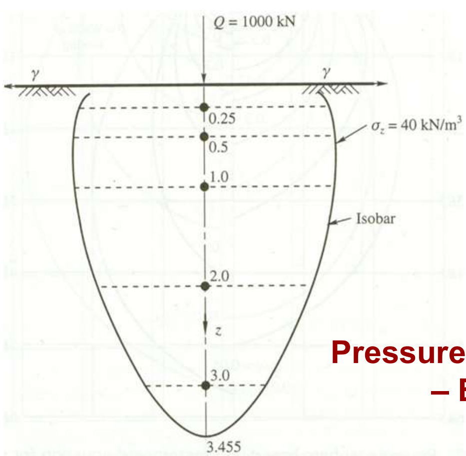
Solution

From Eq. (6.1a) we have

$$\sigma_z = \frac{3Q}{2\pi z^2}\left[\frac{1}{1+(r/z)^2}\right]^{5/2}$$

We may now write by rearranging an equation for the radial distance r as

$$r = \sqrt{z}\sqrt{\left(\frac{3Q}{2\pi z^2\sigma_z}\right)^{2/5} - 1}$$



Pressure Isobars – Example

(Murthy 6.12)